Irrationality of power series with average sparsity

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joint work with
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November 22nd, 2024
Mini Workshop @ TUS

1. Erdős' Irrationality Monsters

Pocket Monsters

In this talk, we let $q \in \mathbb{Z}_{\geq 2}$. The series

$$\sum_{n=1}^{\infty} \frac{\tau(n)}{q^n}, \qquad \sum_{n=1}^{\infty} \frac{r(n)}{q^n}, \qquad \sum_{n=1}^{\infty} \frac{\sigma(n)}{q^n},$$

$$\sum_{n=1}^{\infty} \frac{1}{q^{\varphi(n)}}, \qquad \sum_{n=1}^{\infty} \frac{1}{q^{\sigma(n)}},$$

$$\sum_{n=1}^{\infty} \frac{\sigma(n)}{n!}, \qquad \sum_{n=1}^{\infty} \frac{\sigma_2(n)}{n!}, \qquad \sum_{n=1}^{\infty} \frac{p_n}{n!}$$

are known to be irrational (due to Chowla, Erdős, Duverney, etc.), where

- $\tau(n)$: number of divisors of n.
- $r(n) := \#\{(u, v) \in \mathbb{Z}^2 \mid n = u^2 + v^2\}$.
- $\varphi(n)$: Euler totient function defined by

$$\varphi(n) := n \prod_{p \mid n} \left(1 - \frac{1}{p}\right).$$

- $\sigma_k(n)$: sum of the k-th power of divisors of n, i.e.

$$\sigma_k(n) := \sum_{d \mid n} d^k \quad \text{and write} \quad \sigma(n) := \sigma_1(n).$$

- p_n: the n-th smallest prime number.

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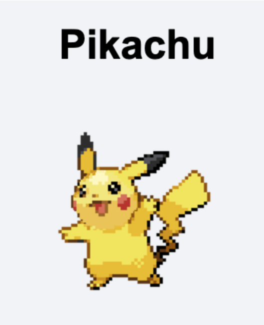

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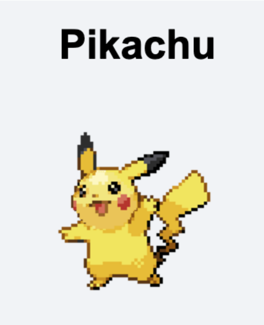
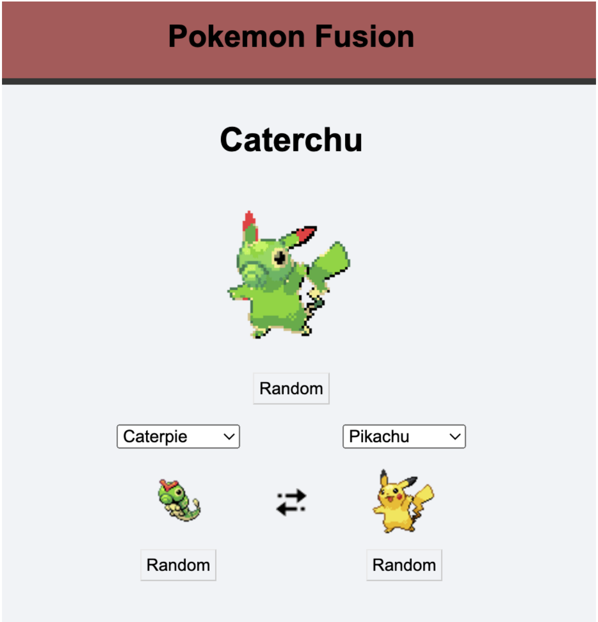


Making chimera


Pokemon Fusion
Caterchu
Random
Caterpie
Pikachu
Random
Random

We now look at two series of Erdős

$$\sum_{n=1}^{\infty} \frac{\tau(n)}{q^n} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{q^{\varphi(n)}}.$$

We shall mix these two series! Namely, we consider

$$\sum_{n=1}^{\infty} \frac{\tau(n)}{q^{\varphi(n)}}.$$

One example of the consequence of our irrationality criterion is:

Theorem 1 (Kaneko–Tachiya–S. (2024+))

$$\sum_{n=1}^{\infty} \frac{\tau(n)}{q^{\varphi(n)}} \notin \mathbb{Q}.$$

Disclaimer

However, our method is purely an extension of the method for

$$\sum_{n=1}^{\infty} \frac{1}{q^{\varphi(n)}}.$$

2. Basic irrationality of sparse base-q expansion

Erdős' series:

$$\xi_\varphi := \sum_{m=1}^{\infty} \frac{1}{q^{\varphi(m)}}.$$

By collecting the terms with the same value of $\varphi(m)$, we get

$$\xi_\varphi = \sum_{n=1}^{\infty} \frac{A_\varphi(n)}{q^n} \quad \text{with} \quad A_\varphi(n) := \sum_{\varphi(m)=n} 1.$$

Thus, our problem is reduced the irrationality of power series

$$\xi := \sum_{n=1}^{\infty} \frac{a(n)}{q^n}.$$

As the simplest case, we consider the base-q expansion

$$\xi := \sum_{n=1}^{\infty} \frac{a(n)}{q^n} \quad \text{and} \quad a(n) \in \{0, 1, \dots, q-1\}.$$

What is special here is the size restriction of a_n.

Notation 1

- For an arithmetic function $a(n)$, we define the **support** of $a(n)$ by

$$\mathrm{Supp}_a := \{n \in \mathbb{N} \mid a(n) \neq 0\}.$$

- For an infinite set $\mathscr{A} \subset \mathbb{N}$ (clear from the context) and $n \in \mathscr{A}$, let

$$n_+ := \min\{m \in \mathscr{A} \mid m > n\} = (\text{the element of } \mathscr{A} \text{ subsequent to } n).$$

Proposition 1 (Long gap in the support implies irrationality)

For any infinitely long base-q expansion

$$\xi = \sum_{n=1}^{\infty} \frac{a(n)}{q^n} \quad \text{with} \quad \#\,\mathrm{Supp}_a = \infty$$

and arbitrary long gaps in the support, i.e.

$$\sup_{n \in \mathrm{Supp}_a} (n_+ - n) = \infty,$$

we have $\xi \notin \mathbb{Q}$.

First averaging technique: Finding long gap on average

In general, it is not easy to locate the exact position of long gap.

We can use the following average argument instead:

Lemma 1 (Finding long gap on average)

For any arithemtic function $a(n)$ with Supp_a, we have

$$\#\,\mathrm{Supp}_a(x) = o(x) \quad (x \to \infty) \implies \sup_{n \in \mathrm{Supp}_a} (n_+ - n) = \infty.$$

Proof.

Take $N \in \mathrm{Supp}_a$ arbitrarily. Then, we have

$$N - \min \mathrm{Supp}_a = \sum_{n \in \mathrm{Supp}_a(N)} (n_+ - n) \leq \#\,\mathrm{Supp}_a(N) \times \sup_{n \in \mathrm{Supp}_a(N)} (n_+ - n)$$

and so, by using $\#\,\mathrm{Supp}_a(x) = o(x) \quad (x \to \infty)$, we get

$$\sup_{n \in \mathrm{Supp}_a(N)} (n_+ - n) \geq \frac{N}{\#\,\mathrm{Supp}_a(N)} + o(1) \to \infty \quad (N \to \infty)$$

as desired. □

Recall our chimera:

$$\xi_\varphi = \sum_{m=1}^{\infty} \frac{1}{q^{\varphi(m)}} = \sum_{n=1}^{\infty} \frac{A_\varphi(n)}{q^n} \quad \text{with} \quad A_\varphi(n) := \sum_{\varphi(m)=n} 1.$$

In order to find long gaps, we need to have

$$\#\,\mathrm{Supp}_{A_\varphi}(x) = o(x) \quad (x \to \infty).$$

To this end, we can use the following:

Lemma 2 (Erdős (1935), Ford (1998))

For the counting function

$$V_\varphi := \{n \in \mathbb{N} \mid \varphi(m) = n \text{ for some } m\} = \mathrm{Supp}_{A_\varphi}$$

(note that $\tau(m) > 0$ for any m), we have

$$\# V_\varphi(x) = \#\,\mathrm{Supp}_{A_\varphi}(x) = \frac{x}{\log x} \exp\Big(O\big((\log\log\log x)^2\big)\Big).$$

3. Erdős' criterion for the irrationality of sparse power series

An issue for the current argument

Indeed, our argument has a flaw: Our method relies on the base-q expansion

$$\xi = \sum_{n=1}^{\infty} \frac{a(n)}{q^n} \quad \text{with} \quad \boxed{a(n) \in \{0, 1, \ldots, q-1\}}$$

while our power series

$$\xi_\varphi = \sum_{m=1}^{\infty} \frac{1}{q^{\varphi(m)}} = \sum_{n=1}^{\infty} \frac{A_\varphi(n)}{q^n} \quad \text{with} \quad A_\varphi(n) := \sum_{\varphi(m)=n} 1$$

indeed have unbounded coefficient!

Proposition 2 (Erdős (1935), Pomerance (1980))

For infinitely many n, we have

$$A_\varphi(n) > n^\theta \quad \text{with} \quad \theta > 0.55655.$$

We again overcome this difficulty with an average argument.

We can indeed replace the size restriction

$$a(n) \in \{0, 1, \ldots, q-1\}$$

by an average boundedness:

Theorem 2 (Erdős' simple criterion (1954))

Consider an infinite power series

$$\xi = \sum_{n=1}^{\infty} \frac{a(n)}{q^n} \quad \text{with} \quad a(n) \in \mathbb{Z}_{\geq 0} \text{ and } \#\,\mathrm{Supp}_a = \infty.$$

Assume

- (Average Bound) "The coefficient is bounded *on average*" in the sense that

$$S_a(x) := \sum_{n<x} a(n) \ll x.$$

- (Average Gap) "We have arbitrary long gaps *on average*" in the sense that

$$\#\,\mathrm{Supp}_a(x) = o(x) \quad (x \to \infty).$$

We then have $\xi \notin \mathbb{Q}$.

For Erdős' series

$$\xi_\varphi = \sum_{m=1}^{\infty} \frac{1}{q^{\varphi(m)}} = \sum_{n=1}^{\infty} \frac{A_\varphi(n)}{q^n} \quad \text{with} \quad A_\varphi(n) := \sum_{\varphi(m)=n} 1,$$

we have average boundedness of coefficient

$$\sum_{n \le x} A_\varphi(n) = \sum_{\varphi(m) \le x} 1 \ll x$$

by the following known result:

Lemma 3 (Erdős (1945), Bateman (1972))

For some constant $c > 0$, we have

$$\sum_{n \le x} A_\varphi(n) = \sum_{\varphi(m) \le x} 1 = \frac{\zeta(2)\zeta(3)}{\zeta(6)} x + O(x \exp(-c(\log x \log\log x)^{\frac{1}{2}})).$$

We now consider the "chimera" series:

$$\xi_\varphi^\tau := \sum_{m=1}^{\infty} \frac{\tau(n)}{q^{\varphi(m)}}.$$

By collecting the terms with the same value of $\varphi(m)$, we get

$$\xi_\varphi^\tau = \sum_{n=1}^{\infty} \frac{A_\varphi^\tau(n)}{q^n} \quad \text{with} \quad A_\varphi^\tau(n) := \sum_{\varphi(m)=n} \tau(m).$$

Can we apply Erdős' simple criterion?

- For the Average Gap condition, we can still use Ford's result

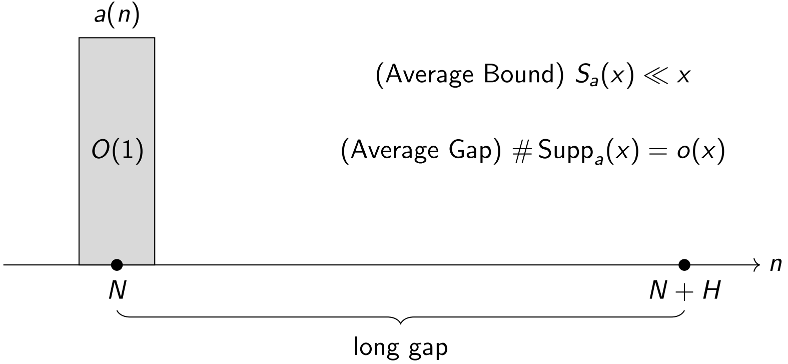
$$\#\,\mathrm{Supp}_{A_\varphi^\tau}(x) = \#V_\varphi(x) = \frac{x}{\log x} \exp\Big(O\big((\log\log\log x)^2\big)\Big).$$

- However, for the Average Bound condition, we can prove

$$\sum_{n<x} A_\varphi^\tau(n) = \sum_{\varphi(m)<x} \tau(m) \sim Cx\log x \quad (x \to \infty)$$

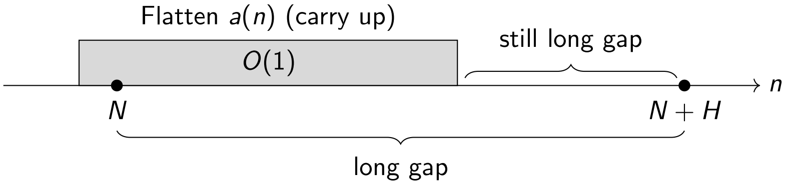
  for some constant $C > 0$. Thus, the Average Bound condition does NOT hold!

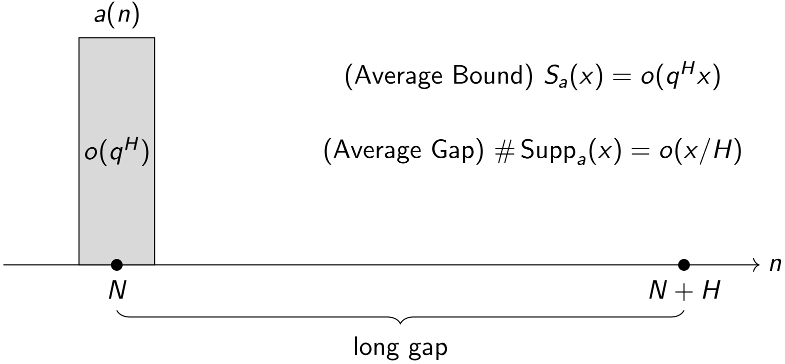
4. A refined Erdős criterion



(Average Bound) $S_a(x) \ll x$

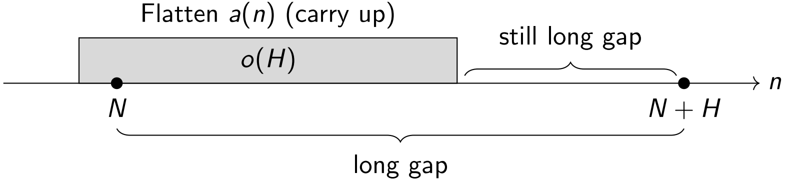
(Average Gap) $\# \,\mathrm{Supp}_a(x) = o(x)$





$$\text{(Average Bound) } S_a(x) = o(q^H x)$$

$$\text{(Average Gap) } \#\,\mathrm{Supp}_a(x) = o(x/H)$$



Theorem 3 (Kaneko–Tachiya–S. (2024+))

Consider an infinite power series

$$\xi = \sum_{n=1}^{\infty} \frac{a(n)}{q^n} \quad \text{with} \quad a(n) \in \mathbb{Z}_{\geq 0} \text{ and } \#\,\mathrm{Supp}_a = \infty.$$

Assume that there is $H\colon \mathbb{N} \to \mathbb{R}_{\geq 1}$ such that

- (Average Bound) The coefficient is bounded by $o(t^{H(x)})$ on average, i.e.

$$S_a(x) := \sum_{n<x} a(n) = o(q^{H(x)}x).$$

- (Average Gap) We have gaps of size $H(x)$ on average, i.e.

$$\#\,\mathrm{Supp}_a(x) = o(x/H(x)) \quad (x \to \infty).$$

- (Strong Convergence) Our power series is strictly inside the disk of convergence, i.e.

$$\limsup_{n\to\infty} a(n)^{\frac{1}{n}} < q.$$

We then have $\xi \notin \mathbb{Q}$. (Remark: We need assumptions only for arbitrary large x.)

Our refined criterion:

$$S_a(x) := \sum_{n<x} a(n) \ll q^{H(x)}x \quad \text{and} \quad \#\,\mathrm{Supp}_a(x) = o\left(\frac{x}{H(x)}\right).$$

The chimera series:

$$\xi_\varphi^\tau = \sum_{n=1}^{\infty} \frac{A_\varphi^\tau(n)}{q^n} \quad \text{with} \quad A_\varphi^\tau(n) := \sum_{\varphi(m)=n} \tau(m).$$

Since

$$\varphi(m) \geq \frac{m}{\log m} \quad (m : \text{large})$$

we have

$$\sum_{n<x} A_\varphi^\tau(n) = \sum_{\varphi(m)<x} \tau(m) \leq \sum_{m<x\log x} \tau(m) \ll x(\log x)^2 \leq q^{\frac{2}{\log q}\log\log x} x = o(q^{H(x)}x)$$

with $H(x) := \frac{3}{\log q}\log\log x$. Also, Ford's result gives (note that $\tau(n) > 0$ for any n)

$$\#\,\mathrm{Supp}_{A_\varphi^\tau}(x) = \#\,\mathrm{Supp}_{A_\varphi}(x) = \frac{x}{\log x}\exp\Big(O\big((\log\log\log x)^2\big)\Big) = o(x/H(x)).$$

Strong convergence is easy to check. This completes the proof of the irrationality of ξ_φ^τ.

5. Proof of the refined criterion

Write

$$\xi = \sum_{n=1}^{\infty} \frac{a(n)}{q^n} = \xi_N + q^{-N} X_N \quad \rightsquigarrow \quad q^N \xi - q^N \xi_N = X_N$$

with

$$\xi_N = \xi_N(a) := \sum_{n<N} \frac{a(n)}{q^n} \quad \text{and} \quad X_N = X_N(a) := \sum_{h \geq 0} \frac{a(N+h)}{q^h}.$$

Lemma 4 (Small tail implies irrationality)

$$\inf_N X_N = 0 \text{ and } \# \operatorname{Supp}_a = \infty \quad \Longrightarrow \quad \xi \notin \mathbb{Q}.$$

Proof.

Assume to the contrary that $\xi = \frac{a}{d}$. Then

$$0 < dX_N = dq^N \xi - dq^N \xi_N = q^N a - d \sum_{n<N} a(n) q^{N-n} \in \mathbb{Z}$$

which implies

$$\inf_N X_N \geq \frac{1}{d},$$

which contradicts the assumption on the infimum. □

The average of the "H-shifted" tail

Our goal:

$$\inf_N X_N = 0 \quad \text{where} \quad X_N = \sum_{h \geq 0} \frac{a(N+h)}{q^h}.$$

We want to make the tail X_N small by making a gap

$$a(N+0) = a(N+1) = \cdots = a(N+H-1) = 0$$

of length H in the way

$$X_N = \sum_{h \geq 0} \frac{a(N+h)}{q^h} = \sum_{h \geq H} \frac{a(N+h)}{q^h}.$$

Again, it is difficult to locate where X_N is small. Thus, we consider an average

$$R(x, H) := \sum_{N < x} \sum_{h \geq H} \frac{a(N+h)}{q^h}$$

of the "H-shifted" tails.

Detecting small tail

Our goal:

$$\inf_N X_N = 0 \quad \text{where} \quad X_N = \sum_{h \geq 0} \frac{a(N+h)}{q^h}.$$

We introduce the counting function for large tails

$$T(x, \varepsilon) := \{N < x \mid X_N \geq \varepsilon\}.$$

Thus, out aim is to bound this quantity non-trivially as

$$\# T(x, \varepsilon) = o(x) \quad (x \to \infty)$$

to show there are plenty of small tails.

Lemma 5 (Small tail lemma)

For any $\eta \in (0,1)$ (a technical parameter), we have the following. Assume

- (Average Gap) We have gaps of size $H(x)$ on average, i.e.

$$\#\,\mathrm{Supp}_a(x) = o(x/H(x)) \quad (x \to \infty).$$

- (Average Small Tail) The $H(x)$-shifted tail is small on average, i.e.

$$R(\eta x, H(x)) = \sum_{N<\eta x} \sum_{h \geq H(x)} \frac{a(N+h)}{q^h} = o(x) \quad (x \to \infty).$$

  (This is not assumed in our criterion and so should be proved later.)

For a fixed $\varepsilon > 0$, we then have

$$\# T(\eta x, \varepsilon) = \#\{N < \eta x \mid X_N \geq \varepsilon\} = o(x) \quad (x \to \infty).$$

We decompose as

$$\begin{aligned}
T(\eta x, \varepsilon) &= \{N < \eta x \mid X_N \geq \varepsilon\} \\
&= \{N < \eta x \mid X_N \geq \varepsilon \text{ and } \forall h \in [0, H(x)),\ a(N+h) = 0\} \\
&\qquad \cup \{N < \eta x \mid X_N \geq \varepsilon \text{ and } \exists h \in [0, H(x)),\ a(N+h) \neq 0\} \\
&= \{N < \eta x \mid X_N \geq \varepsilon \text{ and } \forall h \in [0, H(x)),\ a(N+h) = 0\} \\
&\qquad \cup \{N < \eta x \mid \exists h \in [0, H(x)),\ a(N+h) \neq 0\} \\
&=: T_{\mathrm{Gap}}(\eta x, \varepsilon) \cup T_{\mathrm{No\ Gap}}(\eta x, \varepsilon),
\end{aligned}$$

where

$$\begin{aligned}
T_{\mathrm{Gap}}(\eta x, \varepsilon) &:= \{N < \eta x \mid X_N \geq \varepsilon \text{ and } \forall h \in [0, H(x)),\ a(N+h) = 0\}, \\
T_{\mathrm{No\ Gap}}(\eta x, \varepsilon) &:= \{N < \eta x \mid \exists h \in [0, H(x)),\ a(N+h) \neq 0\}.
\end{aligned}$$

We then have

$$\# T(\eta x, \varepsilon) \leq \# T_{\mathrm{Gap}}(\eta x, \varepsilon) + \# T_{\mathrm{No\ Gap}}(\eta x, \varepsilon).$$

We bound $\# T_{\mathrm{Gap}}(\eta x, \varepsilon)$ and $\# T_{\mathrm{No\ Gap}}(\eta x, \varepsilon)$ separately.

Small tail after a gap

For the gap part

$$T_{\mathsf{Gap}}(\eta x, \varepsilon) = \{N < \eta x \mid X_N \geq \varepsilon \text{ and } \forall h \in [0, H(x)),\ a(N+h) = 0\},$$

by the Average Small Tail Condition, i.e.

$$R(\eta x, H(x)) = \sum_{N < \eta x} \sum_{h \geq H(x)} \frac{a(N+h)}{q^h} = o(x) \quad (x \to \infty),$$

we have

$$\begin{aligned}
\sum_{N \in T_{\mathsf{Gap}}(\eta x, \varepsilon)} X_N &= \sum_{N \in T_{\mathsf{Gap}}(\eta x, \varepsilon)} \sum_{h \geq 0} \frac{a(N+h)}{q^h} \\
&= \sum_{N \in T_{\mathsf{Gap}}(\eta x, \varepsilon)} \sum_{h \geq H(x)} \frac{a(N+h)}{q^h} \\
&\leq \sum_{N < \eta x} \sum_{h \geq H(x)} \frac{a(N+h)}{q^h} = R(\eta x, H(x)) = o(x)
\end{aligned}$$

while

$$\sum_{N \in T_{\mathsf{Gap}}(\eta x, \varepsilon)} X_N \geq \varepsilon \cdot \# T_{\mathsf{Gap}}(\eta x, \varepsilon)$$

and so

$$\# T_{\mathsf{Gap}}(\eta x, \varepsilon) = o(x/\varepsilon) = o(x) \quad (x \to \infty).$$

For the no gap part

$$T_{\text{No Gap}}(\eta x, \varepsilon) = \{N < \eta x \mid \exists h \in [0, H(x)),\ a(N+h) \neq 0\},$$

by Average Gap Condition, i.e.

$$\#\,\mathrm{Supp}_a(x) = o(x/H(x)) \quad (x \to \infty),$$

we have

$$\begin{aligned}
\#T_{\text{No Gap}}(\eta x, \varepsilon) &\leq \#T_{\text{No Gap}}(\eta x - H(x), \varepsilon) + H(x) \\
&\leq \sum_{0 \leq h < H(x)} \#\{N < \eta x - H(x) \mid a(N+h) \neq 0\} + H(x) \\
&\leq \sum_{0 \leq h < H(x)} \#\{N < \eta x \mid a(N) \neq 0\} + H(x) \\
&= H(x)\#\,\mathrm{Supp}_a(\eta x) + H(x) \\
&\leq 2H(x) \cdot \#\,\mathrm{Supp}_a(x) \\
&= o(H(x) \cdot x/H(x)) = o(x).
\end{aligned}$$

This completes the proof of the small tail lemma.

What we need to prove is the Average Small Tail condition, i.e.

$$R(\eta x, H(x)) = \sum_{N<\eta x} \sum_{h \geq H(x)} \frac{a(N+h)}{q^h} = o(x) \quad (x \to \infty)$$

for some $\eta \in (0,1)$, which is not assumed in our criterion.

Lemma 6 (Average small tail lemma)

Assume

- (Average Bound) The coefficient is bounded by $t^{H(x)}$ on average, i.e.

$$S_a(x) = \sum_{n<x} a(n) = o(q^{H(x)} x).$$

- (Strong Convergence) Our power series is strictly inside the disk of convergence, i.e.

$$\limsup_{n \to \infty} a(n)^{\frac{1}{n}} < q.$$

Then, there is $\eta \in (0,1)$ such that

$$R(\eta x, H(x)) = \sum_{N<\eta x} \sum_{h \geq H(x)} \frac{a(N+h)}{q^h} = o(x) \quad (x \to \infty).$$

We decompose the sum as

$$\begin{aligned} R(\eta x, H(x)) &= \sum_{N<\eta x} \sum_{h \geq H(x)} \frac{a(N+h)}{q^h} \\ &= \sum_{N<\eta x} \sum_{H(x) \leq h < x-N} \frac{a(N+h)}{q^h} + \sum_{N<\eta x} \sum_{h \geq x-N} \frac{a(N+h)}{q^h} \\ &=: R_{\mathrm{close}}(\eta x, H(x)) + R_{\mathrm{far}}(\eta x, H(x)). \end{aligned}$$

We then estimate the last two sums separately.

Bound for close tail

For the "close" tail

$$R_{\text{close}}(\eta x, H(x)) = \sum_{N<\eta x} \sum_{H(x)\le h<x-N} \frac{a(N+h)}{q^h},$$

by the Average Bound condition, i.e.

$$S_a(x) = \sum_{n<x} a(n) = o(q^{H(x)}x),$$

we have

$$\begin{aligned} R_{\text{close}}(\eta x, H(x)) &= \sum_{N<\eta x} \sum_{H(x)\le h<x-N} \frac{a(N+h)}{q^h} \\ &\le \sum_{h\ge H(x)} q^{-h} \sum_{N<x-h} a(N+H) \le 2q^{-H(x)} S_a(x) = o(x). \end{aligned}$$

We next consider the "far" tail

$$R_{\text{far}}(\eta x, H(x)) = \sum_{N<\eta x} \sum_{h \geq x-N} \frac{a(N+h)}{q^h}.$$

By the Strong Convergence condition, we can take $\eta \in (0,1)$ such that

$$\limsup_{n\to\infty} a(n)^{\frac{1}{n}} < q^{1-2\eta} < q \quad \rightsquigarrow \quad a(n) \ll q^{(1-2)n}$$

we have

$$\begin{aligned} R_{\text{far}}(x, H(x)) &= \sum_{N<\eta x} \sum_{h \geq x-N} \frac{a(N+h)}{q^h} \\ &\leq \sum_{N<\eta x} q^N \sum_{h \geq x-N} \frac{q^{(1-2\eta)(N+h)}}{q^{N+h}} \\ &\ll_\eta q^{\eta x} \cdot \frac{q^{(1-2\eta)x}}{q^x} \ll q^{-\eta x} = o(1). \end{aligned}$$

This completes the proof of average small tail lemma and so the refined Erdős criterion.

A straightforward application of the refined criterion with

Lemma 7 (Erdős–Hall (1977), Luca–Pomerance (2009))

$$\#\{n < x \mid \varphi(\varphi(m)) = n \text{ for some } m\} = \frac{x}{(\log x)^2} \exp(O(\log\log x \log\log\log x)^{\frac{1}{2}}).$$

(here, the exponent 2 of the denominator $(\log x)^2$ is important) gives

Theorem 4 (Kaneko–Tachiya–S. (2024+))

The series

$$\sum_{n=1}^{\infty} \frac{n^k}{q^{\varphi(\varphi(n))}} \quad (k \in \mathbb{Z}_{\geq 0})$$

are linearly independent over $\mathbb{Q}$.

Conjecture 1

The series

$$\sum_{n=1}^{\infty} \frac{n^k}{q^{\varphi(n)}} \quad (k \in \mathbb{Z}_{\geq 0})$$

are linearly independent over $\mathbb{Q}$.

6. Extension to the higher degree irrationality

Theorem 5 (Erdős (1957))

Let $d \in \mathbb{N}$. For an increasing sequence of positive integers

$$m_1 < m_2 < m_3 < \cdots$$

satisfying

$$\limsup_{k\to\infty} \frac{m_k}{k^d} = \infty,$$

the series

$$\alpha := \sum_{k=1}^{\infty} \frac{1}{q^{m_k}}$$

does not satisfy any non-trivial algebraic equation of degree $\leq d$ over $\mathbb{Q}$, i.e.

$$P \in \mathbb{Q}[X] \setminus \{0\} \text{ and } \deg P \leq d \implies P(\alpha) \neq 0.$$

Expanding the algebraic relation

Write $\mathscr{M} := \{n_k \mid k \in \mathbb{N}\}$. Our series:

$$\alpha := \sum_{m \in \mathscr{M}} \frac{1}{q^m}.$$

Note that we have

$$\alpha^s = \left(\sum_{m \in \mathscr{M}} \frac{1}{q^m} \right)^s = \sum_{n=1}^{\infty} \frac{r_s(n)}{q^n},$$

where the coefficient is given by the representation function

$$r_s(n) := \#\{(m_1, \dots, m_s) \in \mathscr{M}^s \mid m_1 + \cdots + m_s = n\}.$$

For a given polynomial $P(X) = X^d + a_{d-1}X^{d-1} + \cdots + a_1X + a_0$, we expand the expression

$$P(\alpha) = \alpha^d + c_{d-1}\alpha^{d-1} + \cdots + c_1\alpha + a_0$$

to get

$$P(\alpha) = \sum_{n=1}^{\infty} \frac{a(n) + b(n)}{q^n},$$

where the coefficients $a(n), b(n)$ are given by

$$a(n) = r_d(n) \quad \text{and} \quad b(n) = c_{d-1}r_{d-1}(n) + \cdots + c_1r_1(n) + c_0.$$

Remark

- Knight (1991) gave a "direct" proof of the transcendence of the Fredholm series

$$\sum_{k=1}^{\infty} \frac{1}{q^{2^k}}.$$

- Bailey–Borwein–Crandall–Pomerance (2004) used the same idea to discuss the distribution of 1's in the binary expansion and the higher degree irrationality.
- Kaneko (2017) extended the method to the algebraic *independence*.
- We can extend our method to the higher degree independence by Kaneko's method.

Thus, what we need to consider is the irrationality of the series of the type

$$\xi := \sum_{n=1}^{\infty} \frac{a(n)+b(n)}{q^n} \quad \text{with} \quad a(n) \in \mathbb{Z}_{\geq 0},\ \#\operatorname{Supp}_a = \infty \text{ and } b(n) \in \mathbb{Z},$$

where a "perturbation" $b(n)$ is newly inserted into our original series.

Theorem 6 (Erdős (1957))

Assume that there exists $\Delta, L > 1$ satisfying

- (Average Bound) The coefficients are bounded on average, i.e.

$$S_a(x),\ S_{|b|}(x) \ll x.$$

- (Average Gap) We have long gaps on average, i.e.

$$\#\operatorname{Supp}_a(x) = o(x) \quad \text{and} \quad \#\operatorname{Supp}_b(x) = o(x/\log x).$$

- (Strong Convergence) We have $\log a(n), \log|b(n)| \ll n$.
- (Interlace) For any consecutive $n, n_+ \in \operatorname{Supp}_b$ and $\ell \geq L$ with $n + \Delta\ell < n_+$, we have

$$\operatorname{Supp}_a \cap [n+\ell, n+\Delta\ell) \neq \varnothing.$$

Then, we have $\xi \notin \mathbb{Q}$.

Theorem 7 (Erdős (1957) without proof)

Assume that there exists $\Delta, L > 1$ satisfying

- (Average Bound) The coefficients are bounded on average, i.e.

$$S_a(x),\ S_{|b|}(x) \ll x.$$

- (Average Gap) We have long gaps on average, i.e.

$$\#\operatorname{Supp}_a(x), \#\operatorname{Supp}_b(x) = o(x)$$

  (The denominator $\log x$ is removed from the bound of $\#\operatorname{Supp}_b$.)

- (Strong Convergence) We have

$$\limsup_{n\to\infty} a(n)^{\frac{1}{n}},\ \limsup_{n\to\infty} |b(n)|^{\frac{1}{n}} < q.$$

  (This is a weaker bound than the previous one.)

- (Interlace) For any consecutive $n, n_+ \in \operatorname{Supp}_b$ and $\ell \geq L$ with $n + \Delta\ell < n_+$, we have

$$\operatorname{Supp}_a \cap [n+\ell, n+\Delta\ell) \neq \varnothing.$$

Then, we have $\xi \notin \mathbb{Q}$.

Theorem 8 (Kaneko–Tachiya–S. (2024+))

Assume that there exists $H\colon \mathbb{N} \to \mathbb{R}_{\geq 1}$ and $\Delta, L > 1$ satisfying

- (Average Bound) The coefficient is bounded by $o(t^{H(x)})$ on average, i.e.

$$S_a(x), S_{|b|}(x) = o(q^{H(x)}x).$$

- (Average Gap) We have gaps of size $H(x)$ on average, i.e.

$$\#\,\mathrm{Supp}_a(x), \#\,\mathrm{Supp}_b(x) = o(x/H(x)) \quad (x \to \infty).$$

- (Strong Convergence) We have

$$\limsup_{n\to\infty} a(n)^{\frac{1}{n}},\ \limsup_{n\to\infty} |b(n)|^{\frac{1}{n}} < q.$$

- (Interlace) For any consecutive $n, n_+ \in \mathrm{Supp}_b$ and $\ell \geq L$ with $n + \Delta\ell < n_+$, we have

$$\mathrm{Supp}_a \cap [n+\ell, n+\Delta\ell) \neq \varnothing.$$

Then, we have $\xi \notin \mathbb{Q}$.

Together with Kaneko's method to obtain independence, we can get

Theorem 9 (Kaneko–Tachiya–S. (2024+))

For $D \geq 2$ (not including $D = 1$), the series

$$\sum_{n=1}^{\infty} \frac{n^{kD^k}}{q^{\varphi(n)^D}} \quad (k \in \mathbb{Z}_{\geq 0})$$

do not satisfy any non-trivial algebraic relation of degree $\leq D$.