

Almost primes with square digital reverses

Yuta Suzuki
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Landau's problem $\#\{n \in \mathbb{N} \mid n^2 + 1 \text{ prime}\} = \infty$?

$n \in P_r \Leftrightarrow \underbrace{\Omega(n)}_{\text{the number of prime factors of } n \text{ counted with multiplicity}} \leq r$

Partial Result $\#\{n \in \mathbb{N} \mid n^2 + 1 \in P_r\} = \infty$.

World Records

Rademacher (1924) $r=7$

Ricci (1937) $r=5$

Kuhn (1954) $r=3$

Iwaniec (1978) $r=2$

$\xrightarrow{\text{NEXT?}} r=1$ original problem...

GAME SET!

Q. What should we do if a super intelligence ends a game?
(natural/artificial)

A. Start a NEW GAME!
(different?)

Today A variant with digital reverse. e.g.) $\text{rev}(196) = 691$.

$b \in \mathbb{Z}_{\geq 2}$: base

$$n = (n_{L-1} \cdots n_0)_b \rightsquigarrow \text{rev}(n) := (n_0 \cdots n_{L-1})_b.$$

" $\cdot + 1$ " $\rightsquigarrow$ " $\text{rev}(\cdot)$ " in Landau's problem

$\rightsquigarrow$ Conjecture $\# \{ n \in \mathbb{N} \mid \text{rev}(n^2) : \text{prime} \} = \infty$?

Caution! $b=10$. $\text{rev}((40 \cdots 0)^2) = \text{rev}(160 \cdots 0) = 61 \nmid \text{prime!}$

Conjecture $\# \{ p \mid \text{rev}(p) : \text{square} \} = \infty$?

(or $\# \{ n \in \mathbb{N} \mid \text{rev}(n^2) : \text{prime} \ \& \ b \nmid n \} = \infty$?)

Main Thm 1 (S. / ChatGPT 5.6 Sol P_{no} (2026+))

$\left[\forall b \geq 2, \exists n_b \in \mathbb{N} \text{ s.t. } \# \{ n : P_{n_b} \mid \text{rev}(n) : \text{square} \} = \infty \right.$

$\left. \begin{array}{l} \text{originally } b \geq 700 \text{ (S.)} \end{array} \right\}$

$$n = (n_{L-1} \dots n_0)_b \rightsquigarrow \text{rev}_\lambda(n) := (n_0 \dots n_{\lambda-1})_b$$

$$Q_{\lambda, I}(a, g) := \# \{ m \in \underbrace{I}_{\text{interval}} \mid \text{rev}(m^2) \equiv a \pmod{g} \text{ \& } (m, b(b^2-1)) = 1 \}$$

Main Thm 2. (S./ChatGPT (2026)) $x := b^{\frac{\lambda}{2}}$

$\forall b \geq 2, \exists \theta_b > 0$ s.t. $\forall \lambda, \varepsilon \in \mathbb{Z}_{\geq 0}$ with $\lambda \geq 2\varepsilon - 1, \forall I \subset [b^{\frac{\lambda}{2}-1}, b^{\frac{\lambda}{2}})$: interval

$$\sum_{\substack{g \leq Q \\ (g, b(b^2-1))=1}}^{\#} \max_a \left| Q_{\lambda, I}(a, g) - \frac{1}{g} \frac{\varphi(b(b^2-1))}{b(b^2-1)} |I| \right| \ll_{b, \varepsilon} x \exp(-c_b \sqrt{\log x}) \quad \forall \varepsilon > 0$$

provided $Q \leq x^{\theta_b - \varepsilon}$

$\& \text{ "0+2" } \rightsquigarrow \text{ "rev(.)" in Twin Prime}$

cf. Thm (Dartyge-Rivat-Swagrenepoel (2025)).

$$\lfloor \forall b \geq 2, \exists r_b \in \mathbb{N} \text{ s.t. } \# \{ p \mid \text{rev}(p) : P_{r_b} \} = \infty$$

Method Mauduit-Rivat (2018) on the Rudin-Shapiro sequences along squares.
+ DRS (2025).

(We need to use a smaller class of functions.)

(3)

Outline of Proof

orthogonality $\rightarrow$

Goal

$$\sum_{d \asymp b^\delta}^\# \sum_{h(d)}^* |S(\frac{h}{d})| \ll b^{(1-\eta)\delta} \propto (\exists \eta > 0)$$

where

$$S(\alpha) := \sum_{m \in I}^\# e(\alpha \text{rev}_{2^\xi}(m^2)) \quad \text{with } e(\alpha) := \exp(2\pi i \alpha)$$

$m^2 \asymp b^{2\xi}$

orthogonality $\rightarrow$

$$S(\alpha) = \sum_{k(b^{2\xi})} W_\xi(\frac{k}{b^{2\xi}}) F_{2\xi}(\alpha, \frac{k}{b^{2\xi}}) \quad \text{with}$$

$$W_\xi(\beta) := \sum_{0 \leq m < b^\xi} e(\beta m^2)$$

$$F_\lambda(\alpha, \beta) := \frac{1}{b^\lambda} \sum_{0 \leq n < b^\lambda} e(\alpha \text{rev}_\lambda(n) - \beta n)$$

With square root cancellation ...

$$|S(\alpha)| \ll b^{2\xi} \cdot b^{\xi/2} b^{-\xi} = b^{\frac{3}{2}\xi} \quad \text{VERY trivial ...}$$

Obstacle m^2 grows too rapid

Solution (by Mordell-Rivat) van der Corput's differencing ... with rev?!

Key $m \leq b^\xi$, $r < b^p \leq b^\xi$, $\theta > 0$: "margin"

$$\text{rev}_{2\xi}((m+r)^2) - \text{rev}_{2\xi}(m^2) = b^{\xi-(p+\theta)} (\text{rev}_{\xi+p+\theta}((m+r)^2) - \text{rev}_{\xi+p+\theta}(m^2)).$$

(for all but $O(b^{\xi-\frac{1}{2}\theta})$ m's. $m^2 + 2mr + r^2 \ll b^{\xi+p} \implies \dots$)

Why? Carry up does not propagate too long!

With square root cancellation

$$S(\alpha) \ll b^\xi \cdot b^{\xi/2} \cdot b^{-\xi/2} = b^\xi \leftarrow \text{trivial.}$$

the 2nd van der Corput differencing with shifts b^H 's,

which eliminates lower μ digits.

(ChatGPT: No margin " θ " is required for rev in this step.) I overlooked it...

$$S(\alpha) \ll \left(\overline{\Phi}_{\frac{\xi-15p}{4}}(\alpha b^\theta)^{\frac{1}{4}} + b^{-\frac{1}{2}p} \right) \ll (\log x)^{O(1)}.$$

$$\overline{\Phi}_\lambda(\alpha) := \max_{\beta \in \mathbb{R}} |F_\lambda(\alpha, \beta)|^2$$

$$\rightarrow \sum_{d \asymp b^\delta}^\# \sum_{h(d)}^* |S(\frac{b}{d})| \ll \left(\sum_{d \asymp b^\delta}^\# \sum_{h(d)}^* \overline{\Phi}_{\frac{\Sigma-15\rho}{4}}(\frac{b}{d})^{\frac{1}{4}} + \underbrace{b^{2\delta - \frac{1}{2}\rho}}_{\delta \leq \frac{1}{2}\rho} \right) x(\log x)^{O(1)}$$

Lemma (DRS (2025))

$$\left[\sum_{d \asymp b^\delta}^\# \sum_{h(d)}^* \overline{\Phi}_\Delta(\frac{b}{d})^{\frac{1}{4}} \ll b^{(1-\eta)\delta} \quad \text{provided } \delta K_b \delta \leq \Delta \right.$$

$$K(\theta) := \frac{1}{b} \frac{\sin \pi b \theta}{\sin \pi \theta}, \quad b^{-\mathcal{G}_K} := \max_{\alpha \in \mathbb{R}} \frac{1}{b} \sum_{0 \leq l < b} K\left(\frac{1}{b+1} \left\| \frac{\alpha+l}{b} \right\| \right)^K.$$

$$K_b := \min \{ K \in \mathbb{N} \mid \mathcal{G}_K > \frac{1}{2} \}.$$

$$\leadsto \delta \leq \frac{1}{2}\rho \ \& \ \delta K_b \delta \leq \frac{\Sigma-15\rho}{4} \quad \Rightarrow \quad \delta \leq \underbrace{\frac{1}{32K_b+30}}_{\Theta_b} \Sigma$$