THE ROSSER–IWANIEC SIEVE

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Contents

In this note, we shall develop the theory of the Rosser–Iwaniec sieve following [1, Chapter 11], [2, Chapter 4] and [4].

1. Cast of characters – the sieve data –

We first introduce our setting of sieve problem.

Definition 1.1 (Sieve data). A ***sieve data*** is a tuple

$$(\mathcal{A}, \mathcal{P}, z, X, \omega, r)$$

of

- A finite sequence of integers $\mathcal{A}$ called the ***sifting sequence***.
- A set of prime numbers $\mathcal{P}$ called the ***sifting set***.
- A real number $z \geq 2$ called the ***level of sieve***.
- A real number $X > 0$ used as an approximation of $|\mathcal{A}|$.
- A multiplicative function $\omega(d)$ called the ***density function*** satisfying

$$0 \leq \omega(p) < 1 \quad \text{for all prime } p$$

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and

$$p \notin \mathcal{P} \implies \omega(p) = 0. \tag{1.1}$$

- An arithmetic function $r(d)$.

satisfying the local condition

$$\forall d \mid P(z), \quad |\mathcal{A}_d| = \omega(d)X + r(d), \tag{1.2}$$

where $P(z)$ is defined by

$$P(z) = \prod_{\substack{p \in \mathcal{P} \\ p < z}} p$$

and $\mathcal{A}_d$ is the subsequence defined by

$$\mathcal{A}_d = \{a \in \mathcal{A} \mid a \equiv 0 \pmod{d}\} \quad \text{for } d \in \mathbb{N}.$$

As a convention, $|\mathcal{A}_d|$ counts with the multiplicities of elements in $\mathcal{A}$.

Definition 1.2 (Sieve function)**.** For a sieve data

$$(\mathcal{A}, \mathcal{P}, z, X, \omega, r),$$

we define the ***sieve function*** $S(\mathcal{A}, \mathcal{P}, z)$ by

$$S(\mathcal{A}, \mathcal{P}, z) = \sum_{\substack{a \in \mathcal{A} \\ (a, P(z)) = 1}} 1, \tag{1.3}$$

where we count the multiplicities of elements in $\mathcal{A}$.

The main aim of sieve theory is to estimate the sieve function

$$S(\mathcal{A}, \mathcal{P}, z)$$

for a given sieve data $(\mathcal{A}, \mathcal{P}, z, X, \omega, r)$.

2. The Eratosthenes–Legendre sieve

We now see how the setting of Section 1 is used in the development of sieves by reviewing **the Eratosthenes–Legendre sieve**. The basis of this sieve is the well-known formula

$$\sum_{d \mid n} \mu(d) = \begin{cases} 1 & (\text{if } n = 1), \\ 0 & (\text{if } n > 1). \end{cases} \tag{2.1}$$

Theorem 2.1 (The Eratosthenes–Legendre sieve)**.** *For a sieve data*

$$(\mathcal{A}, \mathcal{P}, z, X, \omega, r),$$

we have

$$S(\mathcal{A}, \mathcal{P}, z) = XV(z) + R,$$

where

$$V(z) = \sum_{d \mid P(z)} \mu(d)\omega(d) \quad \text{and} \quad R = \sum_{d \mid P(z)} \mu(d)r(d).$$

Proof. On inserting (2.1) into (1.3), we can calculate $S(\mathcal{A},\mathcal{P},z)$ as

$$S(\mathcal{A},\mathcal{P},z)=\sum_{a\in\mathcal{A}}\sum_{d|(a,P(z))}\mu(d)=\sum_{d|P(z)}\mu(d)|\mathcal{A}_d|.$$

On inserting (1.2) into this equation, we obtain the theorem. □

By recalling (1.1) and expaning the product, we can express $V(z)$ as

$$V(z)=\prod_{p<z}(1-\omega(p)).$$

However, for practical applications, the power of the Eratosthenes–Legendre sieve is limited since the remainder term R is a sum taken over very long range of d.

3. Sieve weights

In order to control the remainder term, we replace the Möbius function $\mu(d)$ by some arithmetic function mimicking $\mu(d)$ with smaller support. By recalling the proof of the Eratosthenes–Legendre sieve, we find that it suffices to consider the arithmetic function defined and mimicking $\mu(d)$ over divisors $d \mid P(z)$. Hence we define lower and upper sieve weights as follows:

Definition 3.1 (Weight data)**.** ***A weight data*** $(\mathcal{P},D,z)$ is a triple of

- A set of primes $\mathcal{P}$.
- Real numbers D and z with $D\geq z\geq 2$.

We call $\mathcal{P}$ the **sifting set** as before, D ***the level of support*** and z **the sifting level**. As in Definition 1.1, we let

$$P(z)=\prod_{\substack{p\in\mathcal{P}\\ p<z}}p$$

for a given weight data.

Definition 3.2 (Sieve weight)**.** For a weight data $(\mathcal{P},D,z)$, two arithmetic functions $\lambda^-(d)$ and $\lambda^+(d)$ defined for $d\mid P(z)$ are called ***a lower bound sieve weight*** and ***an upper bound sieve weight***, respectively, for the weight data $(\mathcal{P},D,z)$, if

(i) The lower and upper bound condition

$$\sum_{d|N}\lambda^-(d)\leq\sum_{d|N}\mu(d)\leq\sum_{d|N}\lambda^+(d) \tag{3.1}$$

holds for every $N\mid P(z)$.

(ii) The support condition

$$d\geq D \text{ and } d\mid P(z) \implies \lambda^{\pm}(d)=0 \tag{3.2}$$

holds.

We prepare the following general overture of sieve machinery.

Lemma 3.3 (Sieve lemma)**.** *Consider*

- *A sieve data* $(\mathcal{A},\mathcal{P},z,X,\omega,r)$.
- *A weight data* $(\mathcal{P},D,z)$.

- *Upper and lower sieve weights* $\lambda^{\pm}(d)$ *for the weight data* $(\mathcal{P}, D, z)$.

Then, we have

$$XV^{-}(z) + R^{-}(D,z) \le S(\mathcal{A},\mathcal{P},z) \le XV^{+}(z) + R^{+}(D,z),$$

where

$$V^{\pm}(z) = V^{\pm}(\mathcal{A},D,z) = \sum_{d|P(z)} \lambda^{\pm}(d)\omega(d),$$

$$R^{\pm}(D,z) = R^{\pm}(\mathcal{A},D,z) = \sum_{\substack{d|P(z)\\ d<D}} \lambda^{\pm}(d)r(d).$$

Proof. By (2.1), as in the proof of the Erastosthenes–Legendre sieve,

$$S(\mathcal{A},\mathcal{P},z) = \sum_{a\in\mathcal{A}} \sum_{d|(a,P(z))} \mu(d).$$

Since $(a, P(z)) \mid P(z)$, Definition 3.2 implies

$$\sum_{a\in\mathcal{A}} \sum_{d|(a,P(z))} \lambda^{-}(d) \le S(\mathcal{A},\mathcal{P},z) \le \sum_{a\in\mathcal{A}} \sum_{d|(a,P(z))} \lambda^{+}(d).$$

By changing the order of summation,

$$\sum_{a\in\mathcal{A}} \sum_{d|(a,P(z))} \lambda^{\pm}(d) = \sum_{d|P(z)} \lambda^{\pm}(d)|\mathcal{A}_d|.$$

By substituting (1.2) and checking the support of $\lambda^{\pm}(d)$, we arrive at the lemma. □

4. The fundamental identity

Let us fix a weight data $(\mathcal{P}, D, z)$ and try to construct a sieve weight for $(\mathcal{P}, D, z)$. Let (ρ_d) be a sequence of complex numbers defined for $d \mid P(z)$. In principle, this sequence is thought as an indicator function of some condition on the variable d and so (ρ_d) is usually defined to take 0 or 1 as its values. In this note, if a square-free integer d with $d \mid P(z)$ is given, we use the following expression

$$d = p_1p_2\cdots p_r, \quad z > p_1 > p_2 > \cdots > p_r, \quad p_1, p_2, \ldots, p_r \in \mathcal{P}, \quad r \ge 0 \tag{4.1}$$

even without special mention. By using this notation, we define

$$\sigma_d \coloneqq \prod_{1\le i\le r} \rho_{p_1\cdots p_i} \text{ if } d \ge 2 \quad\text{and}\quad \sigma_1 \coloneqq 1 \tag{4.2}$$

and

$$\overline{\sigma}_d \coloneqq (1-\rho_{p_1\cdots p_r}) \prod_{1\le i<r} \rho_{p_1\cdots p_i} \text{ if } d \ge 2 \quad\text{and}\quad \overline{\sigma}_1 \coloneqq 0. \tag{4.3}$$

For some sequence (ρ_d) with some superscripts, e.g. $(\rho_d^{\pm})$, we denote the associated σ_d by attaching the same type of superscripts, e.g. $\sigma_d^{\pm}$.

We use the function σ_d for the truncation of Möbius function, i.e. we replace the Möbius function $\mu(d)$ by $\lambda(d) = \mu(d)\sigma_d$. Thus, we consider

$$\sum_{d|n} \mu(d)\sigma_d \tag{4.4}$$

instead of (2.1). The difference between (2.1) and (4.4) can be seen in

$$\sum_{d|n}\mu(d)=\sum_{d|n}\mu(d)\sigma_d+\sum_{d|n}\mu(d)(1-\sigma_d), \tag{4.5}$$

in which we try to "trash" the second term on the right-hand side by using some property of σ_d. In order to use the decomposition (4.5), we take closer look at the "negated indicator function" $(1-\sigma_d)$.

In the following explanation, we think ρ_d or σ_d as indicator functions and we identify the indicator function and the associated condition. We recall that the condition σ_d is the conjunction of the conditions

$$\rho_{p_1},\quad \rho_{p_1p_2},\quad \rho_{p_1p_2p_3},\quad \rho_{p_1p_2p_3p_4},\quad \ldots,\quad \rho_{p_1\cdots p_r},$$

which may be expressed as

$$\rho_{p_1}\wedge\rho_{p_1p_2}\wedge\rho_{p_1p_2p_3}\wedge\rho_{p_1p_2p_3p_4}\wedge\cdots\wedge\rho_{p_1\cdots p_r}.$$

Its negation is, by the de Moivre rule, given by a disjunction

$$(\neg\rho_{p_1})\vee(\neg\rho_{p_1p_2})\vee(\neg\rho_{p_1p_2p_3})\vee(\neg\rho_{p_1p_2p_3p_4})\vee\ldots\vee(\neg\rho_{p_1\cdots p_r}).$$

Let us read these conditions from left, i.e. we call the condition $(\neg\rho_{p_1\cdots p_n})$ the n-th condition. Then we classify the possibilities according to which condition is the first condition failing to hold. Since this classification is disjoint, we can rewrite the above condition by the disjoint disjunction of the conditions

$$\rho_{p_1}\wedge\ldots\wedge\rho_{p_1\cdots p_{n-1}}\wedge(\neg\rho_{p_1\cdots p_n}).$$

Each of the last conditions can be expressed in terms of the indicator function as

$$(1-\rho_{p_1\cdots p_n})\prod_{1\le i<n}\rho_{p_1\cdots p_i}=\overline{\sigma}_{p_1\cdots p_n}.$$

Therefore, in principle, we arrive at the decomposition

$$1-\sigma_d=\sum_{1\le n\le r}\overline{\sigma}_{p_1\cdots p_n}$$

or, by using our convention $\overline{\sigma}_1=0$, we have

$$1-\sigma_d=\sum_{0\le n\le r}\overline{\sigma}_{p_1\cdots p_n}.$$

We introduce two symbols

$$p_{\min}(d)=\min\{p\colon \text{prime factor of } d\},\quad p_{\max}(d)=\max\{p\colon \text{prime factor of } d\}$$

with conventions $p_{\min}(1)=+\infty$ and $p_{\max}(1)=0$. Then, we can write

$$1-\sigma_d=\sum_{\substack{d_1d_2=d\\ p_{\min}(d_1)>p_{\max}(d_2)}}\overline{\sigma}_{d_1}.$$

This is our fundamental decomposition. Our above argument is rather informal nature, so we shall give a more formal proof of the above identity.

Lemma 4.1. *Let $(\mathcal{P},D,z)$ be a weight data. For $d\mid P(z)$, we have*

$$1-\sigma_d=\sum_{\substack{d_1d_2=d\\ p_{\min}(d_1)>p_{\max}(d_2)}}\overline{\sigma}_{d_1}.$$

Proof. By the convention $\overline{\sigma}_1 = 0$, the right-hand side above is

$$\sum_{\substack{d_1 d_2 = d \\ p_{\min}(d_1) > p_{\max}(d_2)}} \overline{\sigma}_{d_1} = \sum_{1 \le n \le r} \overline{\sigma}_{p_1 \cdots p_n}.$$

By recalling the definition (4.3), this is

$$\begin{aligned}
&= \sum_{1 \le n \le r} (1 - \rho_{p_1 \cdots p_n}) \prod_{1 \le i < n} \rho_{p_1 \cdots p_i} \\
&= \sum_{1 \le n \le r} \left(\prod_{1 \le i < n} \rho_{p_1 \cdots p_i} - \prod_{1 \le i \le n} \rho_{p_1 \cdots p_i} \right) \\
&= \sum_{1 \le n \le r} \prod_{1 \le i \le n-1} \rho_{p_1 \cdots p_i} - \sum_{1 \le n \le r} \prod_{1 \le i \le n} \rho_{p_1 \cdots p_i} \\
&= \sum_{0 \le n \le r-1} \prod_{1 \le i \le n} \rho_{p_1 \cdots p_i} - \sum_{1 \le n \le r} \prod_{1 \le i \le n} \rho_{p_1 \cdots p_i} \\
&= 1 - \prod_{1 \le i \le r} \rho_{p_1 \cdots p_i} = 1 - \sigma_d.
\end{aligned}$$

This completes the proof. □

The decomposition given in Lemma 4.1 can be used to prove various identities used in combinatorial sieves. We prepare rather general identity.

Lemma 4.2 (Fundamental identity). *Let $(\mathcal{P}, D, z)$ be a weight data. For any divisor $N \mid P(z)$ and any arithmetic function $f(d)$ defined for $d \mid P(z)$,*

$$\sum_{d \mid N} f(d) \sigma_d = \sum_{d \mid N} f(d) - \sum_{d \mid N} \overline{\sigma}_d \sum_{e \mid (N, P(p_{\min}(d)))} f(de). \tag{4.6}$$

In particular, if $f(d)$ is multiplicative, we have

$$\sum_{d \mid N} f(d) \sigma_d = \sum_{d \mid N} f(d) - \sum_{d \mid N} f(d) \overline{\sigma}_d \sum_{e \mid (N, P(p_{\min}(d)))} f(e). \tag{4.7}$$

Proof. We have

$$\sum_{d \mid N} f(d) \sigma_d = \sum_{d \mid N} f(d) - \sum_{d \mid N} f(d)(1 - \sigma_d).$$

It suffices to consider the second term of the right-hand side. By Lemma 4.1,

$$\begin{aligned}
\sum_{d \mid N} f(d)(1 - \sigma_d) &= \sum_{\substack{d_1 d_2 \mid N \\ p_{\min}(d_1) > p_{\max}(d_2)}} f(d_1 d_2) \overline{\sigma}_{d_1} \\
&= \sum_{d_1 \mid N} \overline{\sigma}_{d_1} \sum_{\substack{d_2 \mid N/d_1 \\ p_{\min}(d_1) > p_{\max}(d_2)}} f(d_1 d_2) \\
&= \sum_{d_1 \mid N} \overline{\sigma}_{d_1} \sum_{d_2 \mid (N, P(p_{\min}(d_1)))} f(d_1 d_2)
\end{aligned}$$

This proves (4.6). By the condition $p_{\min}(d_1) > p_{\max}(d_2)$, we have $(d_1, d_2) = 1$ in the above summation. Thus, if $f(d)$ is multiplicative, we can rewrite $f(d_1 d_2) =$

$\psi(d_1)\psi(d_2)$ and arrive at

$$\sum_{d|N} f(d)(1-\sigma_d) = \sum_{d_1|N} f(d_1)\overline{\sigma}_{d_1} \sum_{d_2|(N,P(p_{\min}(d_1)))} f(d_2).$$

This proves (4.7) and completes the proof. □

5. Preparatory lemmas for combinatorial sieves

As we mentioned, we want to "trash" the second term on the right-hand side of

$$\sum_{d|n} \mu(d) = \sum_{d|n} \mu(d)\sigma_d + \sum_{d|n} \mu(d)(1-\sigma_d).$$

The following lemma carry out such a disposal. Let

$$\nu_+ := 1, \quad \nu_- := 0$$

so that $\mp 1 = (-1)^{\nu_\pm}$.

Lemma 5.1. *Let $(\mathcal{P}, D, z)$ be a weight data and $(\rho_d^\pm)$ be sequences of real numbers defined for $d \mid P(z)$ satisfying the condition*

$$\rho_d^\pm = \begin{cases} 1 & \text{if } \nu(d) \equiv \nu_\mp \pmod 2, \\ 0 \text{ or } 1 & \text{if } \nu(d) \equiv \nu_\pm \pmod 2. \end{cases} \tag{5.1}$$

Then, for every $N \mid P(z)$, we have

$$\sum_{d|N} \mu(d)\sigma_d^- \le \sum_{d|N} \mu(d) \le \sum_{d|N} \mu(d)\sigma_d^+,$$

where $\sigma_d^\pm$ is defined by (4.1), (4.2) and (4.3) with $(\rho_d^\pm)$.

Proof. By taking $f(d) = \mu(d)$ in (4.7), we have

$$\sum_{d|N} \mu(d)\sigma_d^\pm = \sum_{d|N} \mu(d) - \sum_{d|N} \mu(d)\overline{\sigma}_d^\pm \sum_{e|(N,P(p_{\min}(d)))} \mu(e). \tag{5.2}$$

By definition (4.3) and condition (5.1), we have

$$\overline{\sigma}_d^\pm = \begin{cases} 0 & \text{if } \nu(d) \equiv \nu_\mp \pmod 2, \\ 0 \text{ or } 1 & \text{if } \nu(d) \equiv \nu_\pm \pmod 2. \end{cases} \tag{5.3}$$

Returning to (5.2), we have

$$\begin{aligned} \sum_{d|N} \mu(d)\sigma_d^\pm &= \sum_{d|N} \mu(d) - (-1)^{\nu_\pm} \sum_{d|N} \overline{\sigma}_d^\pm \sum_{e|(A,P(p_{\min}(d)))} \mu(e) \\ &= \sum_{d|N} \mu(d) \pm \sum_{d|N} \overline{\sigma}_d^\pm \sum_{e|(N,P(p_{\min}(d)))} \mu(e). \end{aligned}$$

By (2.1), we find that the sum

$$\sum_{d|N} \overline{\sigma}_d^\pm \sum_{e|(N,P(p_{\min}(d)))} \mu(e)$$

is non-negative. Thus the assertion follows. □

As we can see in Lemma 3.3, we need to estimate

$$V^{\pm}(z) = \sum_{d|P(z)} \mu(d)\omega(d)\sigma_d^{\pm}.$$

In particular, we expect that $V^{\pm}(z)$ is rather close to

$$V(z) = \sum_{d|P(z)} \mu(d)\omega(d) = \prod_{p<z}(1-\omega(p)).$$

As for this purpose, we prepare the following lemma.

Lemma 5.2. *Let $g(d)$ be a multiplicative function, $(\mathcal{P}, D, z)$ be a weight data, and $\chi^{\pm}(d)$ be the functions given in* Lemma 5.1. *Then,*

$$V^{\pm}(z) = V(z) \pm \sum_{\substack{d|P(z)\\ \nu(d)\equiv\nu_{\pm} \ (\mathrm{mod}\ 2)}} \omega(d)\overline{\sigma}_d^{\pm} V(p_{\min}(d)).$$

Proof. By using the identity (4.7) with $N = P(z)$ and $f(d) = \mu(d)\omega(d)$, we have

$$\begin{aligned} V^{\pm}(z) &= \sum_{d|P(z)} \mu(d)\omega(d)\sigma_d^{\pm} \\ &= \sum_{d|P(z)} \mu(d)\omega(d) - \sum_{d|P(z)} \mu(d)\omega(d)\overline{\sigma}_d^{\pm} \sum_{e|P(p_{\min}(d))} \mu(e)\omega(e) \\ &= V(z) - \sum_{d|P(z)} \mu(d)\omega(d)\overline{\sigma}_d^{\pm} V(p_{\min}(d)) \end{aligned}$$

By recalling (5.3), this is

$$\begin{aligned} &= V(z) - \sum_{\substack{d|P(z)\\ \nu(d)\equiv\nu_{\pm} \ (\mathrm{mod}\ 2)}} \mu(d)\omega(d)\overline{\sigma}_d^{\pm} V(p_{\min}(d)) \\ &= V(z) \pm \sum_{\substack{d|P(z)\\ \nu(d)\equiv\nu_{\pm} \ (\mathrm{mod}\ 2)}} \omega(d)\overline{\sigma}_d^{\pm} V(p_{\min}(d)). \end{aligned}$$

This completes the proof. □

6. Rosser's weight

In this note, we use Rosser's construction of combinatorial sieves. In order to develop his method, we need the notion of *sieve dimension*.

For a given weight data $(\mathcal{P}, D, z)$, let us introduce the **sifting variable**

$$s := \frac{\log D}{\log z}$$

for the logarithmic scale of the level of support relative to the level of sieve. In the remaining part of this note, we take a parameter $\beta \geq 1$ and assume

$$s \geq \beta \geq 1.$$

Then, we use the sequence $(\rho_d^{\pm})$ defined by

$$\rho_{p_1\cdots p_r}^{\pm} = \begin{cases} 0 & \text{if } p_1 \cdots p_r \cdot p_r^{\beta} \geq D \text{ and } r \equiv \nu_{\pm} \ (\mathrm{mod}\ 2), \\ 1 & \text{otherwise.} \end{cases} \tag{6.1}$$

Then, we immediately find that our $\mu(d)\sigma_d^{\pm}$ gives a sieve weight for $(\mathcal{P}, D, z)$.

Lemma 6.1. *Let $(\mathcal{P}, D, z)$ be a weight data, β be a real number with $s \geq \beta \geq 1$, and define $\sigma_d^{\pm}$ by the sequence (6.1). Then,*

$$\mu(d)\sigma_d^- \quad \text{and} \quad \mu(d)\sigma_d^+$$

are lower and uppwer bound sieve weight for $(\mathcal{P}, D, z)$, respectively.

Proof. The upper and lower bound conditions (3.1) has been already proven in Lemma 5.1. Therefore, it suffices to prove the support condition (3.2). We prove the contraposition $\sigma_d^{\pm} \neq 0 \Rightarrow d < D$. Assume $\sigma_d^{\pm} \neq 0$ and write d as

$$d = p_1 p_2 \cdots p_r, \quad z > p_1 > p_2 > \cdots > p_r, \quad p_1, p_2, \ldots, p_r \in \mathcal{P}, \quad r \geq 0.$$

If $r = 0$, then we have $d = 1 < D$. If $r = 1$, then since $d \mid P(z)$, we have

$$d = p_1 < z = D^{\frac{1}{s}} \leq D.$$

If $r \geq 2$ and $r \equiv \nu_{\pm} \pmod 2$, then we have $\rho_d^{\pm} \neq 0$ so that

$$d = p_1 \cdots p_r \leq p_1 \cdots p_r^{\beta+1} < D.$$

If $r \geq 2$ and $r \not\equiv \nu_{\pm} \pmod 2$, then $r - 1 \equiv \nu_{\pm} \pmod 2$ and $\rho^{\pm}_{p_1 \cdots p_{r-1}} \neq 0$ so that

$$d = p_1 \cdots p_r \leq p_1 \cdots p_{r-1}^{\beta+1} < D$$

since $p_{r-1} > p_r$ and $\beta \geq 1$. This completes the proof. □

The next task is to approximate

$$V^{\pm}(z) = \sum_{d \mid P(z)} \mu(d)\omega(d)\sigma_d^{\pm}$$

by the original

$$V(z) = \sum_{d \mid P(z)} \mu(d)\omega(d)$$

From now on, we assume that the following data is given:

- A fixed sifting set $\mathcal{P}$.
- A fixed level of sieve D.
- A fixed real numbers $\kappa \geq 0$ and $K > 1$.
- A fixed density function $\omega \in \Omega(\mathcal{P}, \kappa, K)$.
- Rosser's weight $\mu(d)\sigma_d^{\pm}$ defined by (6.1) with a real parameter $\beta \geq 1$.

unless otherwise specified.

Lemma 6.2. *We have*

$$V^{\pm}(z) = V(z) \pm \sum_{\substack{n \geq 1 \\ n \equiv \nu_{\pm} \pmod 2}} V_n(z),$$

where

$$\begin{aligned} V_n(z) &= V_n(D, z) \\ &:= \sum_{\substack{z > p_1 > \cdots > p_n \\ p_1 \cdots p_m p_m^{\beta} < D \ (1 \leq m < n,\, m \equiv n \pmod 2) \\ p_1 \cdots p_n p_n^{\beta} \geq D}} \omega(p_1 \cdots p_n) V(p_n). \end{aligned} \tag{6.2}$$

Proof. By Lemma 5.2 and the convention $\overline{\sigma}_1 = 0$, it suffices to show

$$\sum_{\substack{d|P(z)\\ \nu(d)\equiv\nu_\pm \ (\mathrm{mod}\ 2)}} \omega(d)\overline{\sigma}_d^{\pm}V(p_{\min}(d)) = \sum_{\substack{n\ge1\\ n\equiv\nu_\pm \ (\mathrm{mod}\ 2)}} V_n(z).$$

We first classify the terms by the value of $\nu(d)$ as

$$\sum_{\substack{d|P(z)\\ \nu(d)\equiv\nu_\pm \ (\mathrm{mod}\ 2)}} \omega(d)\overline{\sigma}_d^{\pm}V(p_{\min}(d)) = \sum_{\substack{n=1\\ n\equiv\nu_\pm \ (\mathrm{mod}\ 2)}}^{\infty} \sum_{\substack{d|P(z)\\ \nu(d)=n}} \omega(d)\overline{\sigma}_d^{\pm}V(p_{\min}(d)).$$

By recalling the definition of $\overline{\sigma}_d^{\pm}$, we find

$$n \equiv \nu_\pm \ (\mathrm{mod}\ 2) \implies \sum_{\substack{d|P(z)\\ \nu(d)=n}} \omega(d)\overline{\sigma}_d^{\pm}V(p_{\min}(d)) = V_n(z).$$

Thus the lemma follows. □

Lemma 6.3. *We have* $n \le s-\beta \implies V_n(z) = 0$.

Proof. If the sum in (6.2) is non-empty, then there is $(p_1,\ldots,p_n)$ satisfying

$$z > p_1 > \cdots > p_n$$
$$p_1\cdots p_m \cdot p_m^{\beta} < D \ (1 \le m < n,\ m \equiv n \,(\mathrm{mod}\ 2))$$
$$p_1\cdots p_n \cdot p_n^{\beta} \ge D$$

Then, the first and the third condition give

$$z^s = D \le p_1\cdots p_n^{\beta+1} < z^{\beta+n}$$

so that $n > s-\beta$. Thus, if $n \le s-\beta$, then the sum in (6.2) becomes an empty sum so that $V_n(z) = 0$. This completes the proof. □

7. Recurrence relation for $V_n(z)$

The main goal of the remaining part of this note is to give more satisfactory analysis on the Rosser–Iwaniec sieve. The aim of this section is to derive some recurrence formula of $V_n(z)$. Recall $V_n(z)$ defined for a positive integer n by

$$(6.2) \qquad V_n(z) = V_n(D,z) = \sum_{\substack{z>p_1>\cdots>p_n\\ p_1\cdots p_m p_m^{\beta}<D \ (1\le m<n,\, m\equiv n \,(\mathrm{mod}\ 2))\\ p_1\cdots p_n p_n^{\beta}\ge D}} \omega(p_1\cdots p_n)V(p_n)$$

as in (6.2). In order to describe the result, for a positive integer n, we introduce

$$y_n := D^{\frac{1}{\beta+n}}, \quad z_n(s) := \min(D^{\frac{1}{s}}, D^{\frac{1}{\beta+\varepsilon_n}}), \quad \varepsilon_n := \begin{cases} 0 & (\text{if } n \text{ is even}),\\ 1 & (\text{if } n \text{ is odd}).\end{cases}$$

Lemma 7.1. *For $D \geq 1$, $z \geq 2$, $s := \frac{\log D}{\log z}$ and $n \in \mathbb{N}$, we have*

$$V_n(D,z) = \begin{cases} \displaystyle\sum_{y_1 \leq p < z} \omega(p) V(p) & \text{if } n = 1, \\ \displaystyle\sum_{y_n \leq p < z_n} \omega(p) V_{n-1}\left(\frac{D}{p}, p\right) & \text{if } n \geq 2 \text{ and } s \geq \beta - \varepsilon_n. \end{cases}$$

Proof. We first consider the case $n = 1$. By definition,

$$V_1(D,z) = \sum_{\substack{z > p_1 \\ p_1^{\beta+1} \geq D}} \omega(p_1) V(p_1). \tag{7.1}$$

The second summation condition can be rewritten as

$$p_1^{\beta+1} \geq D \iff p_1 \geq D^{\frac{1}{\beta+1}} = y_1.$$

Then (7.1) now gives

$$V_1(D,z) = \sum_{y_1 \leq p_1 < z} \omega(p_1) V(p_1).$$

This completes the proof for the case $n = 1$.

We next consider the case $n \geq 2$ and n is even. First, we remark that we may introduce the condition $p_1 \geq y_n$ into the summation on the right-hand side of

$$V_n(D,z) = \sum_{\substack{z > p_1 > \cdots > p_n \\ p_1 \cdots p_m p_m^\beta < D \ (1 \leq m < n,\, m \equiv n \ (\mathrm{mod}\ 2)) \\ p_1 \cdots p_n p_n^\beta \geq D}} \omega(p_1 \cdots p_n) V(p_n)$$

since the original summation condition implies

$$D \leq p_1 \cdots p_n p_n^\beta \leq p_1^{\beta+n} \quad \text{so that} \quad p_1 \geq D^{\frac{1}{\beta+n}} = y_n.$$

Thus, we have

$$V_n(D,z) = \sum_{\substack{y_n \leq p_1 < z \\ p_1 > \cdots > p_n \\ p_1 \cdots p_m p_m^\beta < D \ (1 \leq m < n,\, m \equiv n \ (\mathrm{mod}\ 2)) \\ p_1 \cdots p_n p_n^\beta \geq D}} \omega(p_1 \cdots p_n) V(p_n). \tag{7.2}$$

Then, we again rewrite the summation condition on the right-hand side. Since n is even, the third condition is rewritten as

$$\begin{aligned} & p_1 \cdots p_m p_m^\beta < D \quad (1 \leq m < n,\ m \equiv n \ (\mathrm{mod}\ 2)) \\ \iff & p_1 \cdots p_m p_m^\beta < D \quad (2 \leq m < n,\ m \equiv n \ (\mathrm{mod}\ 2)) \\ \iff & p_2 \cdots p_m p_m^\beta < D/p_1 \quad (2 \leq m < n,\ m \equiv n \ (\mathrm{mod}\ 2)), \end{aligned}$$

and the fourth condition is rewritten as

$$p_1 \cdots p_n p_n^\beta \geq D \iff p_2 \cdots p_n p_n^\beta \geq D/p_1.$$

Therefore, by (7.2),

$$\begin{aligned} V_n(D,z) &= \sum_{\substack{y_n \le p_1 < z \\ p_1 > \cdots > p_n \\ p_2 \cdots p_m p_m^{\beta} < D/p_1 \ (2 \le m < n,\, m \equiv n \ (\mathrm{mod}\ 2)) \\ p_2 \cdots p_n p_n^{\beta} \ge D/p_1}} \omega(p_1 \cdots p_n) V(p_n) \\ &= \sum_{y_n \le p_1 < z} \omega(p_1) \sum_{\substack{p_1 > p_2 > \cdots > p_n \\ p_2 \cdots p_m p_m^{\beta} < D/p_1 \ (2 \le m < n,\, m \equiv n \ (\mathrm{mod}\ 2)) \\ p_2 \cdots p_n p_n^{\beta} \ge D/p_1}} \omega(p_2 \cdots p_n) V(p_n) \\ &= \sum_{y_n \le p_1 < z} \omega(p_1) V_{n-1}\left(\frac{D}{p_1}, p_1\right). \end{aligned}$$

By assuming $s \ge \beta = \beta - \varepsilon_n$, we have

$$z_n(s) = \min(D^{\frac{1}{s}}, D^{\frac{1}{\beta+\varepsilon_n}}) = \min(D^{\frac{1}{s}}, D^{\frac{1}{\beta}}) = D^{\frac{1}{s}} = z.$$

Thus we obtain the assertion for the case $n \ge 2$ and n is even.

We finally consider the case $n \ge 2$ and n is odd. We can again write

$$\begin{aligned} V_n(D,z) &= \sum_{\substack{z > p_1 > \cdots > p_n \\ p_1 \cdots p_m p_m^{\beta} < D \ (1 \le m < n,\, m \equiv n \ (\mathrm{mod}\ 2)) \\ p_1 \cdots p_n p_n^{\beta} \ge D}} \omega(p_1 \cdots p_n) V(p_n) \\ &= \sum_{\substack{y_n \le p_1 < z \\ p_1 > \cdots > p_n \\ p_1 \cdots p_m p_m^{\beta} < D \ (1 \le m < n,\ m \equiv n \ (\mathrm{mod}\ 2)) \\ p_1 \cdots p_n p_n^{\beta} \ge D}} \omega(p_1 \cdots p_n) V(p_n). \end{aligned}$$

We then rewrite the third condition as

$$\begin{aligned} & p_1 \cdots p_m p_m^{\beta} < D \quad (1 \le m < n,\ m \equiv n \ (\mathrm{mod}\ 2)) \\ & \iff p_1 \cdots p_m p_m^{\beta} < D \quad (2 \le m < n,\ m \equiv n \ (\mathrm{mod}\ 2)) \quad \text{and} \quad p_1^{\beta+1} < D \\ & \iff p_2 \cdots p_m p_m^{\beta} < D/p_1 \quad (2 \le m < n,\ m \equiv n \ (\mathrm{mod}\ 2)) \quad \text{and} \quad p_1 < D^{\frac{1}{\beta+1}}. \end{aligned}$$

Also, we rewrite the fourth condition as

$$p_1 \cdots p_n p_n^{\beta} \ge D \quad \iff \quad p_2 \cdots p_n p_n^{\beta} \ge D/p_1.$$

Therefore, we have

$$\begin{aligned} V_n(D,z) &= \sum_{\substack{y_n \le p_1 < z_n \\ p_1 > \cdots > p_n \\ p_2 \cdots p_m p_m^{\beta} < D/p_1 \ (2 \le m < n,\, m \equiv n \ (\mathrm{mod}\ 2)) \\ p_2 \cdots p_n p_n^{\beta} \ge D/p_1}} \omega(p_1 \cdots p_n) V(p_n) \\ &= \sum_{y_n \le p_1 < z_n} \omega(p_1) \sum_{\substack{p_1 > p_2 > \cdots > p_n \\ p_2 \cdots p_m p_m^{\beta} < D/p_1 \ (2 \le m < n,\, m \equiv n \ (\mathrm{mod}\ 2)) \\ p_2 \cdots p_n p_n^{\beta} \ge D/p_1}} \omega(p_2 \cdots p_n) V(p_n) \\ &= \sum_{y_n \le p_1 < z_n} \omega(p_1) V_{n-1}\left(\frac{D}{p_1}, p_1\right). \end{aligned}$$

This completes the proof. □

8. Partial summation with $\omega(p)V(p)$

We apply partial summation to the recurrence equation obtained in Section 7. For this purpose, we prepare some lemmas on partial summation with

$$\omega(p)V(p).$$

To this end, we introduce the following requirement on $\omega(d)$:

Definition 8.1 (Density function)**.** A multiplicative function $\omega(d)$ satisfying

$$0 \leq \omega(p) < 1 \quad \text{for all prime } p$$

is called a ***density function***. We denote the set of all density function by Ω. For a set of primes $\mathcal{P}$ and a density function $\omega(d)$, if the condition

$$p \notin \mathcal{P} \implies \omega(p) = 0$$

holds, then we say that $\omega(d)$ is supported on $\mathcal{P}$. We denote the set of all density function supported on $\mathcal{P}$ by $\Omega(\mathcal{P})$.

Definition 8.2 (Sieve dimension)**.** We say that a density function $\omega(d)$ has the **sieve dimension** $\kappa > 0$ with constant $K \geq 2$ if

$$\frac{V(w)}{V(z)} = \prod_{w \leq p < z} (1 - \omega(p))^{-1} \leq \left(1 + \frac{K}{\log w}\right)\left(\frac{\log z}{\log w}\right)^{\kappa} \quad \text{for all } z \geq w \geq 2.$$

We denote the set of all density functions of sieve dimension $\kappa > 0$ supported on $\mathcal{P}$ with constant $K \geq 2$ by $\Omega(\kappa, K) = \Omega(\mathcal{P}, \kappa, K)$.

Lemma 8.3. *For $z \geq 1$, we have*

$$V(z) = 1 - \sum_{p < z} \omega(p)V(p).$$

Proof. We have

$$V(z) = \sum_{d | P(z)} \mu(d)\omega(p) = 1 + \sum_{\substack{d | P(z) \\ d > 1}} \mu(d)\omega(d).$$

We next classify d in the last sum by the value of $p_{\max}(d)$. Then,

$$\begin{aligned} V(z) &= 1 + \sum_{p < z} \sum_{\substack{d | P(z) \\ p_{\max}(d) = p}} \mu(d)\omega(d) \\ &= 1 - \sum_{p < z} \omega(p) \sum_{d | P(p)} \mu(d)\omega(d) = 1 - \sum_{p < z} \omega(p)V(p). \end{aligned}$$

This completes the proof. □

Lemma 8.4. *For $z \geq w \geq 1$, we have*

$$\sum_{w \leq p < z} \omega(p) \frac{V(p)}{V(z)} = \frac{V(w)}{V(z)} - 1.$$

Proof. By using Lemma 8.3 with $z := z$ and $z := w$ and taking their difference,

$$V(w) - V(z) = -\sum_{p < w} \omega(p) V(p) + \sum_{p < z} \omega(p) V(p) = \sum_{w \leq p < z} \omega(p) V(p).$$

By dividing both sides by $V(z)$, we obtain

$$\frac{V(w)}{V(z)} - 1 = \sum_{w \leq p < z} \omega(p) \frac{V(p)}{V(z)}.$$

This completes the proof. □

Lemma 8.5. *Consider*

- *Real numbers $z, w, D \geq 2$ with $w \geq z$ and write $z = D^{\frac{1}{s}}, w = D^{\frac{1}{\sigma}}$.*
- *A real-valued continuous function $H(t)$ on $t \in [s, \sigma]$.*

Also, define a function $E(w, z)$ by

$$\frac{V(w)}{V(z)} = \left(\frac{\log z}{\log w} \right)^{\kappa} + E(w, z) \quad \text{for } z \geq w \geq 2.$$

Then, we have

$$\sum_{w \leq p < z} \omega(p) \frac{V(p)}{V(z)} H\left(\frac{\log D}{\log p} \right)$$
$$= \int_s^{\sigma} H(t) \frac{dt^{\kappa}}{s^{\kappa}} + E(w, z) H(\sigma) + \int_w^z E(x, z) dH\left(\frac{\log D}{\log x} \right).$$

Proof. By Lemma 8.4 and partial summation,

$$\sum_{w \leq p < z} \omega(p) \frac{V(p)}{V(z)} H\left(\frac{\log D}{\log p} \right) = -\int_w^z H\left(\frac{\log D}{\log x} \right) d\left(\sum_{x \leq p < z} \omega(p) \frac{V(p)}{V(z)} \right)$$
$$= -\int_w^z H\left(\frac{\log D}{\log x} \right) d\left(\frac{V(x)}{V(z)} \right).$$

By integrating by parts,

$$\sum_{w \leq p < z} \omega(p) \frac{V(p)}{V(z)} H\left(\frac{\log D}{\log p} \right)$$
$$= -H(s) + \frac{V(w)}{V(z)} H(\sigma) + \int_w^z \frac{V(x)}{V(z)} dH\left(\frac{\log D}{\log x} \right).$$

By recalling the definition of $E(w, z)$, we have

$$\sum_{w \leq p < z} \omega(p) \frac{V(p)}{V(z)} H\left(\frac{\log D}{\log p} \right) = I(w, z) + E_1(w, z),$$

where

$$I(w,z) := -H(s) + \left(\frac{\log z}{\log w}\right)^{\kappa} H(\sigma) + \int_w^z \left(\frac{\log z}{\log x}\right)^{\kappa} dH\left(\frac{\log D}{\log x}\right)$$

$$E_1(w,z) := E(w,z)H(\sigma) + \int_w^z E(x,z)dH\left(\frac{\log D}{\log x}\right).$$

For $I(w,z)$, we use integration by parts to obtain

$$I(w,z) = -\int_w^z H\left(\frac{\log D}{\log x}\right) d\left(\frac{\log z}{\log x}\right)^{\kappa}.$$

By changing the variable via

$$\frac{\log D}{\log x} = t,$$

we arrive at

$$I(w,z) = -\left(\frac{\log z}{\log D}\right)^{\kappa} \int_w^z H\left(\frac{\log D}{\log x}\right) d\left(\frac{\log D}{\log x}\right)^{\kappa} = \int_s^{\sigma} H(t)\frac{dt^{\kappa}}{s^{\kappa}}.$$

This completes the proof. □

Lemma 8.6. *Consider*

- *Real numbers $z, w, D \geq 2$ with $w \geq z$ and write $z = D^{\frac{1}{s}}, w = D^{\frac{1}{\sigma}}$.*
- *A real-valued continuous function $H(t)$ non-negative on $t \in [s,\sigma]$.*

Assume $\omega \in \Omega(\kappa, K)$ and $H(t)t^{\kappa}$ is non-increasing for $t \in [s,\sigma]$. Then we have

$$\sum_{w \leq p < z} \omega(p)\frac{V(p)}{V(z)} H\left(\frac{\log D}{\log p}\right) \leq \int_s^{\sigma} H(t)\frac{dt^{\kappa}}{s^{\kappa}} + \frac{(\kappa+1)KH(s)}{\log w}.$$

Proof. Since $\omega \in \Omega(\kappa, K)$, under the notation of Lemma 8.5, we have

$$E(w,z) \leq \frac{K}{\log w}\left(\frac{\log z}{\log w}\right)^{\kappa} = \frac{K}{\log z}\left(\frac{\log z}{\log w}\right)^{\kappa+1}. \tag{8.1}$$

By Lemma 8.5, it suffices to show

$$E_1 := E(w,z)H(\sigma) + \int_w^z E(x,z)dH\left(\frac{\log D}{\log x}\right) \leq \frac{(\kappa+1)KH(s)}{\log w}.$$

By the assumption, $H(t)t^{\kappa}$ in non-increasing. Thus $H(t)$ itself is also non-increasing. Hence, by substituting (8.1) with using the positivity and monotonicity of H,

$$E_1 \leq \frac{K}{\log z}\left(\frac{\log z}{\log w}\right)^{\kappa+1} H(\sigma) + \frac{K}{\log z}\int_w^z \left(\frac{\log z}{\log x}\right)^{\kappa+1} dH\left(\frac{\log D}{\log x}\right).$$

By integration by parts, we have

$$E_1 \leq \frac{KH(s)}{\log z} - \frac{K}{\log z}\int_w^z H\left(\frac{\log D}{\log x}\right) d\left(\frac{\log z}{\log x}\right)^{\kappa+1}.$$

Then we change the variable via

$$\frac{\log D}{\log x} = t.$$

This gives

$$E_1 \le \frac{KH(s)}{\log z} + \frac{K}{\log z}\int_s^\sigma H(t)\frac{dt^{\kappa+1}}{s^{\kappa+1}} = \frac{KH(s)}{\log z} + \frac{(\kappa+1)K}{\log z}\int_s^\sigma H(t)t^\kappa \frac{dt}{s^{\kappa+1}}.$$

Since the function $H(t)t^\kappa$ is non-increasing,

$$\begin{aligned} E_1 &= \frac{KH(s)}{\log z} + \frac{(\kappa+1)KH(s)}{\log z}\left(\frac{\sigma - s}{s}\right) \\ &\le \frac{(\kappa+1)KH(s)}{\log z}\left(\frac{\sigma}{s}\right) = \frac{(\kappa+1)KH(s)}{\log w}. \end{aligned}$$

This completes the proof. □

For a technical reason we face later, we modify Lemma 8.6 so as that the upper endpoint is not necessarily to be z, the variable of the denominator $V(z)$.

Lemma 8.7. *Consider*

- *Real numbers $z, v, w, D \ge 2$ with $z \ge v \ge w$ and write*
$$z = D^{\frac{1}{s}}, v = D^{\frac{1}{\tau}}, w = D^{\frac{1}{\sigma}}.$$
- *A real-valued continuous function $H(t)$ non-negative on $t \in (\tau, \sigma]$.*

Assume $\omega \in \Omega(\kappa, K)$ and $H(t)t^\kappa$ is non-increasing for $t \in (\tau, \sigma]$. Then we have

$$\sum_{w \le p < v} \omega(p)\frac{V(p)}{V(z)}H\left(\frac{\log D}{\log p}\right) \le \int_\tau^\sigma H(t)\frac{dt^\kappa}{s^\kappa} + \frac{3(\kappa+1)K^2H(\tau)}{\log w}\left(\frac{\tau}{s}\right)^\kappa.$$

Proof. By Lemma 8.6 with $z := v$, we have

$$\begin{aligned} \sum_{w \le p < v} \omega(p)\frac{V(p)}{V(z)}H\left(\frac{\log D}{\log p}\right) &= \frac{V(v)}{V(z)}\sum_{w \le p < v} \omega(p)\frac{V(p)}{V(v)}H\left(\frac{\log D}{\log p}\right) \\ &\le \frac{V(v)}{V(z)}\left(\int_\tau^\sigma H(t)\frac{dt^\kappa}{\tau^\kappa} + \frac{(\kappa+1)KH(\tau)}{\log w}\right). \end{aligned}$$

Since $\omega \in \Omega(\kappa, K)$, we have

$$\frac{V(v)}{V(z)} \le \left(\frac{\tau}{s}\right)^\kappa \left(1 + \frac{K}{\log v}\right).$$

Therefore, we have

$$\begin{aligned} &\sum_{w \le p < v} \omega(p)\frac{V(p)}{V(z)}H\left(\frac{\log D}{\log p}\right) \\ &\le \left(1 + \frac{K}{\log v}\right)\left(\int_\tau^\sigma H(t)\frac{dt^\kappa}{s^\kappa} + \frac{(\kappa+1)KH(\tau)}{\log w}\left(\frac{\tau}{s}\right)^\kappa\right) \\ &\le \int_\tau^\sigma H(t)\frac{dt^\kappa}{s^\kappa} + \frac{K}{\log v}\int_\tau^\sigma H(t)\frac{dt^\kappa}{s^\kappa} + \left(1 + \frac{K}{\log v}\right)\frac{(\kappa+1)KH(\tau)}{\log w}\left(\frac{\tau}{s}\right)^\kappa. \end{aligned}$$

By the monotonicity of $H(t)t^{\kappa-1}$, we have

$$\frac{K}{\log v}\int_\tau^\sigma H(t)\frac{dt^\kappa}{s^\kappa} = \frac{\kappa K}{\log v}\int_\tau^\sigma H(t)t^{\kappa-1}\frac{dt}{s^\kappa}$$

$$\le \frac{\kappa K H(\tau)}{\log v}\left(\frac{\tau}{s}\right)^{\kappa}\left(\frac{\sigma-\tau}{\tau}\right)$$
$$\le \frac{\kappa K H(\tau)}{\log w}\left(\frac{\tau}{s}\right)^{\kappa}.$$

Also, we have

$$1+\frac{K}{\log v}\le 1+\frac{K}{\log 2}\le 1+\frac{3}{2}K\le 2K$$

since $K\ge 2$. Therefore, we have

$$\frac{K}{\log v}\int_{\tau}^{\sigma} H(t)\frac{dt^{\kappa}}{s^{\kappa}}+\left(1+\frac{K}{\log v}\right)\frac{(\kappa+1)KH(\tau)}{\log w}\left(\frac{\tau}{s}\right)^{\kappa}$$
$$\le \frac{\kappa KH(\tau)}{\log w}\left(\frac{\tau}{s}\right)^{\kappa}+\frac{2(\kappa+1)K^2H(\tau)}{\log w}\left(\frac{\tau}{s}\right)^{\kappa}\le \frac{3(\kappa+1)K^2H(\tau)}{\log w}\left(\frac{\tau}{s}\right)^{\kappa}.$$

This completes the proof. □

9. Heuristic approximation of $V_n(D,z)$

We now apply Lemma 8.6 heuristically to guess the behavior of our sum

$$V_n(D,z).$$

We recall the recurrence formula

$$V_n(D,z)=\begin{cases}\displaystyle\sum_{y_1\le p<z}\omega(p)V(p) & \text{if } n=1,\\ \displaystyle\sum_{y_n\le p<z_n}\omega(p)V_{n-1}\left(\frac{D}{p},p\right) & \text{if } n\ge 2 \text{ and } s\ge\beta-\varepsilon_n.\end{cases}$$

given in Lemma 7.1. We shall approximate $V_n(D,z)$ in the form

$$V_n(D,z)\approx V(z)f_n(s)$$

with suitable $f_n(s)$. For $n=1$, Lemma 8.5 with $H(t)=1$ implies

$$V_1(D,z)=V(z)\sum_{y_1\le p<z}\omega(p)\frac{V(p)}{V(z)}\approx V(z)\int_{(s,\beta+1]}\frac{dt^{\kappa}}{s^{\kappa}}$$

by ignoring the error term, where the integral is thought to be zero if the integration range is an empty set. Therefore, our first function $f_1(s)$ should be defined as

$$s^{\kappa}f_1(s)=\int_{(s,\beta+1]}dt^{\kappa}.$$

For general n, we use

$$V_{n-1}(D,z)\approx V(z)f_{n-1}(s)$$

as "the induction hypothesis" and Lemma 8.5 with $H(t)=f_{n-1}(t-1)$ to get

$$V_n(D,z)=\sum_{y_n\le p<z_n}\omega(p)V_{n-1}\left(\frac{D}{p},p\right)$$
$$\approx V(z)\sum_{y_n\le p<z_n}\omega(p)\frac{V(p)}{V(z)}f_{n-1}\left(\frac{\log D}{\log p}-1\right)$$
$$\approx V(z)\int_{(\max(s,\beta+\varepsilon_n),\beta+n]}f_{n-1}(t-1)\frac{dt^{\kappa}}{s^{\kappa}}\quad \text{for } s\ge\beta-\varepsilon_n.$$

Therefore, $f_n(s)$ should be defined as

$$s^\kappa f_n(s) = \int_{(\max(s,\beta+\varepsilon_n),\beta+n]} f_{n-1}(t-1)dt^\kappa \quad \text{for } s \geq \beta - \varepsilon_n.$$

According to the above observations, we define functions $f_n(s)$ as follows.

We define a sequence of continuous functions

$$f_1(s),\ f_2(s),\ f_3(s),\ \ldots$$

recursively as follows. We first prepare the intervals

$$I_+ := (\beta - 1, +\infty) \quad \text{and} \quad I_- := [\beta, +\infty)$$

and

$$I_n = I_n(\beta) := \begin{cases} I_+ & \text{if } n \text{ is odd,} \\ I_- & \text{if } n \text{ is even.} \end{cases}$$

The function $f_n(s)$ will be defined and continuous on I_n. Note that

$$I_n \subseteq (0, +\infty) \quad \text{for all } n \geq 1$$

since $\beta \geq 1$. The initial function $f_1(s)$ is defined on I_1 by

$$s^\kappa f_1(s) := \int_{(s,\beta+1]} dt^\kappa, \tag{9.1}$$

where in what follows, the integration over empty interval is thought to be zero. For $n \geq 2$, the function $f_n(s)$ is defined on I_n by the recursion

$$s^\kappa f_n(s) := \int_{(\max(s,\beta+\varepsilon_n),\beta+n]} f_{n-1}(t-1)dt^\kappa \tag{9.2}$$

Our final choice of β will satisfy

$$\beta > 1 \quad \text{if } \kappa > \tfrac{1}{2} \tag{9.3}$$

and so we assume this condition. Under this assumption, the above definition of $(f_n)_{n=1}^\infty$ is well-defined.

Proposition 9.1.
Functions $(f_n)_{n=1}^\infty$ *are well-defined by* (9.1) *and* (9.2) *provided* (9.3).

Proof. Since

$$I_n \subseteq (0, +\infty),$$

the division by s^κ in (9.1), (9.2) are legitimate. We prove the assertion and

$$f_n(s) \ll s^{-\kappa} \quad \text{for } s \in I_n \cap (0, \beta] \text{ and odd } n \geq 1 \tag{9.4}$$

with the implicit constant independent of s by the induction on n.

Initial case. For the case $n = 1$, (9.1) has no problem and we have

$$f_1(s) \leq \left(\frac{\beta+1}{s}\right)^\kappa$$

for $s \in I_1 \cap (0, \beta]$ and so (9.4) holds.

Recursion step from f_{n-1} to f_n with even $n \geq 2$. We should check the integrand $f_{n-1}(t-1)$ in (9.2) is already defined in the previous step and the integral has a finite value. In the integral of (9.2), we have

$$t - 1 > \max(s, \beta + \varepsilon_n) - 1 = \max(s, \beta) - 1 \geq \beta - 1.$$

Thus, $f_{n-1}(t-1)$ is already defined in the previous step. We next check the integral

$$\int_{(\max(s,\beta+\varepsilon_n),\beta+n]} f_{n-1}(t-1)dt^\kappa = \int_{(\max(s,\beta),\beta+n]} f_{n-1}(t-1)dt^\kappa$$

of (9.2) is finite. Since $f_{n-1}(t)$ is continuous on I_{n-1}, it suffices to check the integral

$$\int_\beta^{\beta+1} f_{n-1}(t-1)dt^\kappa$$

around β is finite. By (9.4), if $\beta > 1$, we can simply check the finiteness as

$$0 \leq \int_\beta^{\beta+1} f_{n-1}(t-1)dt^\kappa \ll \int_\beta^{\beta+1} \frac{dt^\kappa}{(t-1)^\kappa} \leq \left(\frac{\beta+1}{\beta-1}\right)^\kappa < +\infty$$

If $\beta = 1$, by (9.4), we can still check the finiteness as

$$0 \leq \int_\beta^{\beta+1} f_{n-1}(t-1)dt^\kappa \ll \int_1^2 \frac{t^{\kappa-1}}{(t-1)^\kappa} dt < +\infty$$

since $\kappa \leq \frac{1}{2}$ if $\beta = 1$ by (9.3). Thus, $f_n(s)$ is well-defined.

Recursion step from f_{n-1} to f_n with odd $n \geq 2$. We should check the integrand $f_{n-1}(t-1)$ in (9.2) is already defined in the previous step, the integral has a finite value and also the bound (9.4). In the integral of (9.2), we have

$$t - 1 \geq \max(s, \beta + \varepsilon_n) - 1 = \max(s, \beta + 1) - 1 \geq \beta.$$

Thus, $f_{n-1}(t-1)$ is already defined in the previous step. Since $f_{n-1}(t-1)$ is continuous in the integration range including the end points, the integral in (9.2) is finite. By (9.2) for the previous step, $f_{n-1}(s)$ is non-negative and so

$$s^\kappa f_n(s) \leq \int_{\beta+1}^{\beta+n} f_{n-1}(t-1)dt^\kappa < +\infty$$

for $s \in I_n \cap (0, \beta]$. This shows the bound (9.4). □

Under the assumption (9.3), we now successfully defined functions $f_n(s)$ by

$$\begin{cases} s^\kappa f_1(s) := \int_{(s,\beta+1]} dt^\kappa, \\ s^\kappa f_n(s) := \int_{(\max(s,\beta+\varepsilon_n),\beta+n]} f_{n-1}(t-1)dt^\kappa \quad \text{for } n \geq 2, \end{cases}$$

where $f_n(s)$ is defined on I_n. We next derive some basic properties of $f_n(s)$.

Proposition 9.2. *Assume* (9.3). *We then have*

(i) *For $n \geq 1$ and $s \geq \beta + n$, we have $f_n(s) = 0$.*
(ii) *For $n \geq 1$, the function $f_n(s)$ is continuous and non-negative on $s \in I_n$.*
(iii) *For $n \geq 1$, the functions $s^\kappa f_n(s)$ and $f_n(s)$ are non-increasing on $s \in I_n$.*

(iv) *For $n \geq 2$, we have*

$$s^\kappa f_n(s) = \int_{\max(s,\beta+\varepsilon_n)}^{\infty} f_{n-1}(t-1)dt^\kappa \quad \text{for } s \in I_n.$$

(v) *For $n \geq 2$, the function $f_n(s)$ is of class C^1 on $(\beta+\varepsilon_n, +\infty)$.*
(vi) *For odd $n \geq 3$, the function $f_n(s)$ is constant for $\beta - 1 < s \leq \beta + 1$.*

Proof.
(i) Immediate from the definition.
(ii), (iii) Immediate from the definition by using induction.
(iv) Easily follows by (i) proven above.
(v) Immediate from (iv) proven above and the continuity of $f_n(s)$.
(vi) Immediate from the definition. □

By the above obtained heuristics, we may expect

$$\begin{aligned} V^\pm(D,z) &= V(z) \pm \sum_{\substack{n\geq 1 \\ n\equiv \nu_\pm \ (\mathrm{mod}\ 2)}} V_n(D,z) \\ &\approx V(z)\Bigg(1 \pm \sum_{\substack{n\geq 1 \\ n\equiv \nu_\pm \ (\mathrm{mod}\ 2)}} f_n(s)\Bigg) \end{aligned}$$

provided the series

$$T^\pm(s) \coloneqq \sum_{\substack{n\geq 1 \\ n\equiv \nu_\pm \ (\mathrm{mod}\ 2)}} f_n(s) \tag{9.5}$$

are convergent. We shall prove this convergence later for $s \in I_\pm$ with β in some appropriate range. Assuming the series $T^\pm(s)$ converges, we define

$$F^\pm(s) \coloneqq 1 \pm T^\pm(s)$$

so that our heuristic approximation will be

$$V^\pm(D,z) \approx V(z)F^\pm(s).$$

We give some observations on $T^\pm(s)$ assuming the convergence of (9.5) for $s \in I_\pm$.

We start with the partial sum

$$T_N(s) \coloneqq \sum_{\substack{1\leq n\leq N \\ n\equiv N \ (\mathrm{mod}\ 2)}} f_n(s) \quad \text{for } N \in \mathbb{N}. \tag{9.6}$$

We have the following result parallel to $T^\pm(s)$.

Proposition 9.3.
(i) *For $N \geq 1$, the function $T_N(s)$ is continuous and non-negative on I_N.*
(ii) *For $N \geq 1$, the function $s^\kappa T_N(s)$ and $T_N(s)$ are decreasing and*

$$T_N^\pm(s) = 0 \quad \text{for } s \geq \beta + N.$$

(iii) *We have*

$$(s^\kappa T_N(s))' = -\kappa s^{\kappa-1} T_{N-1}(s-1) \quad \text{for } s > \beta + \varepsilon_N.$$

(iv) *For odd N, we have*

$$s^\kappa T_N(s) = A_N - s^\kappa \quad \text{for } \beta - 1 < s \le \beta + 1,$$

where

$$A_N = A_N(\kappa, \beta) := (\beta+1)^\kappa T_N(\beta+1) + (\beta+1)^\kappa.$$

(v) *For even N, we have*

$$s^\kappa T_N(s) = -B_N + s^\kappa - A_{N-1} \int_\beta^s \frac{dt^\kappa}{(t-1)^\kappa} \quad \text{for } \beta \le s \le \beta + 2,$$

where A_{N-1} is given as in (iv) *and $B_N = B_N(\beta, \kappa)$ is determined by*

$$\beta^\kappa T_N(\beta) = \beta^\kappa - B_N.$$

Proof.

(i), (ii) Immediately follows by Proposition 9.2.

(iii) For odd N, it suffices to consider the range $s > \beta + 1$. We then have $f_1(s) = 0$. Therefore, by the definition of $f_n(s)$, we have

$$\begin{aligned}
(s^\kappa T_N(s))' &= \frac{d}{ds}\Bigg(\sum_{\substack{1 \le n \le N \\ n \equiv 1 \pmod 2}} s^\kappa f_n(s) \Bigg) \\
&= \frac{d}{ds}\Bigg(\sum_{\substack{3 \le n \le N \\ n \equiv 1 \pmod 2}} s^\kappa f_n(s) \Bigg) \\
&= -\kappa s^{\kappa-1} \sum_{\substack{3 \le n \le N \\ n \equiv 1 \pmod 2}} f_{n-1}(s-1) \\
&= -\kappa s^{\kappa-1} \sum_{\substack{2 \le n \le N-1 \\ n \equiv 0 \pmod 2}} f_n(s-1) = -\kappa s^{\kappa-1} T_{N-1}(s-1).
\end{aligned}$$

For even N, we consider the range $s > \beta$. We similarly have

$$\begin{aligned}
(s^\kappa T_N(s))' &= \frac{d}{ds}\Bigg(\sum_{\substack{2 \le n \le N \\ n \equiv 0 \pmod 2}} s^\kappa f_n(s) \Bigg) \\
&= -\kappa s^{\kappa-1} \sum_{\substack{2 \le n \le N \\ n \equiv 0 \pmod 2}} f_{n-1}(s-1) \\
&= -\kappa s^{\kappa-1} \sum_{\substack{1 \le n \le N-1 \\ n \equiv 1 \pmod 2}} f_n(s-1) = -\kappa s^{\kappa-1} T_{N-1}(s-1).
\end{aligned}$$

This proves (iii).

(iv) For $\beta - 1 < s \le \beta + 1$, by Proposition 9.2 and the definition of $f_1(s)$, we have

$$s^\kappa T_N(s) = \sum_{\substack{3 < n \le N \\ n \equiv 1 \pmod 2}} s^\kappa f_n(s) + f_1(s)$$

$$= \sum_{\substack{3<n\le N \\ n\equiv 1 \pmod 2}} (\beta+1)^\kappa f_n(\beta+1) + (\beta+1)^\kappa - s^\kappa$$
$$= (\beta+1)^\kappa T_N(\beta+1) + (\beta+1)^\kappa - s^\kappa = A_N - s^\kappa.$$

Therefore, (iv) holds.

(v) By the continuity of $T_N(s)$ at $s=\beta$ from right and (iii), we have

$$s^\kappa T_N(s) = \beta^\kappa T_N(\beta) - \int_\beta^s T_{N-1}(t-1)dt^\kappa \quad \text{for } \beta \le s \le \beta+2.$$

By (iv), we then have

$$s^\kappa T_N(s) = \beta^\kappa - B_N - A_{N-1}\int_\beta^s \frac{dt^\kappa}{(t-1)^\kappa} + \int_\beta^s dt^\kappa$$
$$= -B_N + s^\kappa - A_{N-1}\int_\beta^s \frac{dt^\kappa}{(t-1)^\kappa} \quad \text{for } \beta \le s \le \beta+2.$$

This proves (v). □

We next consider the full series $T^\pm(s)$. For convenience, we introduce

$$\varepsilon_+ := 1 \quad \text{and} \quad \varepsilon_- := 0.$$

Proposition 9.4. *Assume the series $T^\pm(s)$ converge for $s \in I_\pm$. Then,*

(i) *The convergence of the series $T^\pm(s)$ are compactly uniform on $I_\pm$.*

(ii) *The functions $T^\pm(s)$ are positive and continuous on I_n.*

(iii) *The series $T^\pm(s)$ can be differentiated term by term on $(\beta+\varepsilon_\pm, +\infty)$.*

(iv) *The functions $T^\pm(s)$ are continuously differentiable on $(\beta+\varepsilon_\pm, +\infty)$.*

(v) *The functions $T^\pm(s), s^\kappa T^\pm(s)$ are non-increasing and*

$$\lim_{s\to\infty} s^\kappa T^\pm(s) = 0.$$

(vi) *We have*

$$(s^\kappa T^\pm(s))' = -\kappa s^{\kappa-1} T^\mp(s-1) \quad \text{for } s > \beta+\varepsilon_\pm.$$

(vii) *We have*

$$s^\kappa T^+(s) = A - s^\kappa \quad \text{for } \beta - 1 < s \le \beta+1,$$

where

$$A = A(\kappa,\beta) := (\beta+1)^\kappa T^+(\beta+1) + (\beta+1)^\kappa.$$

(viii) *We have*

$$s^\kappa T^-(s) = -B + s^\kappa - A\int_\beta^s \frac{dt^\kappa}{(t-1)^\kappa} \quad \text{for } \beta \le s \le \beta+2,$$

where A is given as in (vii) *and $B = B(\beta,\kappa)$ is determined by*

$$\beta^\kappa T^-(\beta) = \beta^\kappa - B.$$

(*This strange notation is motivated by the equation $\beta^\kappa F^-(\beta) = B$.*)

Proof.
(i) Take $s_0 \in I_\pm$. By the non-negativity and monotonicity of $f_n(s)$, the series

$$f_n(s_0) \quad \text{with } n \geq 1 \text{ with } n \equiv \nu_\pm \pmod 2$$

is a majorizing sequence of

$$f_n(s) \quad \text{with } n \geq 1 \text{ with } n \equiv \nu_\pm \pmod 2$$

for $s \geq s_0$. Then, the result follows by the Weierstrass M-test.
(ii) Follows from (ii) of Proposition 9.2 and (i) proven above.
(iii), (iv) It suffices to prove the series

$$\sum_{\substack{n \geq 3 \\ n \equiv 1 \pmod 2}} f_n(s)$$

can be differentiated term by term for $s > \beta + 1$ and

$$\sum_{\substack{n \geq 2 \\ n \equiv 0 \pmod 2}} f_n(s)$$

can be differentiated term by term for $s > \beta$. Note that the functions $f_n(s)$ in these series are of class C^1 on the associated range $I_\pm$ by Proposition 9.2. The point-wise convergence of these series are assumed in (9.5). Furthermore, by using the definition of $f_n(s)$, the term-wise differentiated series are

$$\kappa s^{\kappa-1} \sum_{\substack{n \geq 3 \\ n \equiv 1 \pmod 2}} f_{n-1}(s-1) = -\kappa s^{\kappa-1} \sum_{\substack{n \geq 2 \\ n \equiv 0 \pmod 2}} f_n(s-1),$$

$$\kappa s^{\kappa-1} \sum_{\substack{n \geq 2 \\ n \equiv 0 \pmod 2}} f_{n-1}(s-1) = -\kappa s^{\kappa-1} \sum_{\substack{n \geq 1 \\ n \equiv 1 \pmod 2}} f_n(s-1)$$

which converge compact uniformly by (i) above. By these conditions, we can justify the term-by-term differentiation and the continuity of the differentiated series.
(v) Follows by (i), (ii) and (iii) of Proposition 9.2.
(vi), (vii), (viii) Take the limit $N \to \infty$ in (iii), (iv) and (v) of Proposition 9.3. □

10. Delay differential equation

We next study the functions

$$T^\pm(s) := \sum_{\substack{n \geq 1 \\ n \equiv \nu_\pm \pmod 2}} f_n(s).$$

conditionally defined assuming the convergence. By Proposition 9.4, these functions are indeed a solution of the system of the delay-differential equations

$$(s^\kappa T^\pm(s))' = -\kappa s^{\kappa-1} T^\mp(s-1) \quad \text{for } s > \beta + \varepsilon_\pm \tag{10.1}$$

with initial conditions

$$s^\kappa T^+(s) = A - s^\kappa \quad \text{for } \beta - 1 < s \leq \beta + 1,$$
$$\beta^\kappa T^-(\beta) = \beta^\kappa - B$$

with some real numbers A, B. In Proposition 9.4, we have also seen

$$s^\kappa T^-(s) = -B + s^\kappa - A\int_\beta^s \frac{dt^\kappa}{(t-1)^\kappa} \quad \text{for } \beta \le s \le \beta+1.$$

In order to deal with the system of equations (10.1), we introduce

$$\begin{cases} P(s) := F^+(s) + F^-(s) = T^+(s) - T^-(s) + 2 \\ Q(s) := F^+(s) - F^-(s) = T^+(s) + T^-(s) \end{cases} \quad \text{for } s \ge \beta. \tag{10.2}$$

Then, the above equation (10.1) implies

$$\begin{cases} (s^\kappa P(s))' = +\kappa s^{\kappa-1} P(s-1) \\ (s^\kappa Q(s))' = -\kappa s^{\kappa-1} Q(s-1) \end{cases} \quad \text{for } s > \beta+1. \tag{10.3}$$

The effect of considering the linear combinations P, Q instead of $T^\pm$ can be seen in this equation (10.3): it eliminates the alternating feature of the delay-differential equation (10.1). Also, we have

$$\begin{cases} s^\kappa P(s) = A + B + A\displaystyle\int_\beta^s \frac{dt^\kappa}{(t-1)^\kappa} \\ s^\kappa Q(s) = A - B - A\displaystyle\int_\beta^s \frac{dt^\kappa}{(t-1)^\kappa} \end{cases} \quad \text{for } \beta \le s \le \beta+1. \tag{10.4}$$

By requiring these equations hold in the range $(\beta, \beta+1)$, we extend $P(s), Q(s)$ by

$$s^\kappa P(s) = s^\kappa Q(s) = A \quad \text{for } \beta - 1 < s < \beta. \tag{10.5}$$

Note that for the extended $P(s), Q(s)$, the equation (10.3) can be written as

$$\begin{cases} (s^\kappa P(s))' = +\kappa s^{\kappa-1} P(s-1) \\ (s^\kappa Q(s))' = -\kappa s^{\kappa-1} Q(s-1) \end{cases} \quad \text{for } s \in (\beta, \beta+1) \cup (\beta+1, +\infty).$$

or, equivalently,

$$\begin{cases} sP'(s) = -\kappa P(s) + \kappa P(s-1) \\ sQ'(s) = -\kappa Q(s) - \kappa Q(s-1) \end{cases} \quad \text{for } s \in (\beta, \beta+1) \cup (\beta+1, +\infty). \tag{10.6}$$

In this section, we study the solutions of such delay-differential equations. Note that the extended part (10.5) of $P(s), Q(s)$ are not related to $T^\pm(s)$ by (10.2).

10.1. **Delay differential equation.** For $a, b, C, D \in \mathbb{R}$ and $\beta \ge 1$, we consider the delay differential equation of the form

$$sR'(s) + aR(s) + bR(s-1) = 0 \quad \text{for } s \in (\beta, \beta+1) \cup (\beta+1, +\infty) \tag{10.7}$$

where the solution

$$R\colon (\beta - 1, +\infty) \to \mathbb{R}$$

is assumed to be

(R1) The solution $R(s)$ is continuous on $[\beta, +\infty)$.
(R2) The solution $R(s)$ is differentiable on $(\beta, \beta+1) \cup (\beta+1, +\infty)$.
(R3) The solution $R(s)$ is locally integrable on $(\beta - 1, +\infty)$.

with an initial function

$$s^a R(s) = C - Ds^a \quad \text{for } \beta - 1 < s < \beta. \tag{10.8}$$

Consider the following sets of solutions R of (10.7):

$$\begin{aligned}&\mathsf{DDE}(a,b,\beta)\\&\coloneqq \{R\colon (\beta-1,+\infty)\to\mathbb{R} \mid R \text{ satisfies (R1), (R2) and (10.7)}\},\\&\mathsf{DDE}(a,b,\beta,C,D)\\&\coloneqq \{R\colon (\beta-1,+\infty)\to\mathbb{R} \mid R \text{ satisfies (R1), (R2), (10.7) and (10.8)}\}.\end{aligned}$$

10.2. **Adjoint equation.** The **adjoint equation** of (10.7) is given by

$$(sr(s))' = ar(s) + br(s+1) \quad \text{for } s \in (0,+\infty) \tag{10.9}$$

where the solution $r(s)$ is assumed to be defined and of class C^1 on $(0,+\infty)$. For $R \in \mathsf{DDE}(a,b,\beta)$ and $r(s)$ satisfying (10.9), define their ***Iwaniec pairing*** by

$$\langle R,r\rangle(s) = \langle R,r\rangle_b(s) \coloneqq sr(s)R(s) - b\int_{s-1}^{s} r(t+1)R(t)dt \quad \text{for } s > \beta,$$

where the integral on the right-hand side exists by (R3). By the continuity of $r(s)$ on $(0,+\infty)$, the continuity of $R(s)$ on $[\beta,+\infty)$ and the initial value condition (10.8), we find that $\langle R,r\rangle(s)$ is continuous for $s>\beta$.

We use the solution of adjoint equation (any one of the solutions works) to study the behavior of the given solution $R(s)$ of the original delay differential equation. The key property of the solution of the adjoint equation is the following.

Lemma 10.1. *For $R \in \mathsf{DDE}(a,b,\beta)$ and a solution of r of (10.9),*

$$\langle R,r\rangle(s) \coloneqq sr(s)R(s) - b\int_{s-1}^{s} r(t+1)R(t)dt$$

is a constant function for $s>\beta$.

Proof. For $s \in (\beta,\beta+1)\cup(\beta+1,+\infty)$, by taking the derivative, we obtain

$$\frac{d\langle R,r\rangle(s)}{ds} = (sr(s))'R(s) + sr(s)R'(s) - br(s+1)R(s) + br(s)R(s-1).$$

By (10.7) and (10.9), we have

$$\begin{aligned}\frac{d\langle R,r\rangle(s)}{ds} &= (ar(s)+br(s+1))R(s) + r(s)(-aR(s) - bR(s-1))\\&\qquad - br(s+1)R(s) + br(s)R(s-1)\\&= ar(s)R(s) - ar(s)R(s) = 0.\end{aligned}$$

Thus, the pairing $\langle R,r\rangle(s)$ is a constant on $(\beta,\beta+1)$ and $(\beta+1,+\infty)$. Then, the continuity of $\langle R,r\rangle(s)$ at $s=\beta+1$, we obtain the result. □

In order to use the key property

$$\langle R,r\rangle(s) = sr(s)R(s) - b\int_{s-1}^{s} r(t+1)R(t)dt = (\text{constant}),$$

we need a tool to calculate the constant on the right-hand side. For this purpose, we can use the next lemma.

Lemma 10.2. *For $R \in \mathsf{DDE}(a,b,\beta,C,D)$ and a solution of r of (10.9), we have*

$$\begin{aligned}&\langle R,r\rangle(s)\\&=\beta r(\beta)(R(\beta)-C\beta^{-a})+C\lim_{\sigma\searrow\beta}r(\sigma-1)(\sigma-1)^{1-a}+bD\int_\beta^{\beta+1}r(t)dt\end{aligned}$$

for $s>\beta$.

Proof. By Lemma 10.1, we have

$$\langle R,r\rangle(s)=\lim_{\sigma\searrow\beta}\langle R,r\rangle(\sigma).$$

We obviously have

$$\lim_{\sigma\searrow\beta}\sigma r(\sigma)R(\sigma)=\beta r(\beta)R(\beta)$$

since $R(s)$ is continuous at $\sigma=\beta$ from right. We also have

$$-b\int_{\sigma-1}^{\sigma}r(t+1)R(t)dt=-b\int_{\sigma-1}^{\beta}r(t+1)R(t)dt+o(1)\quad\text{as }\sigma\searrow\beta.$$

By (10.8) and (10.9), we have

$$\begin{aligned}&-b\int_{\sigma-1}^{\beta}r(t+1)R(t)dt\\&=C\int_{\sigma-1}^{\beta}\frac{(-br(t+1))}{t^a}dt+bD\int_{\sigma-1}^{\beta}r(t+1)dt\\&=C\int_{\sigma-1}^{\beta}\frac{ar(t)-(tr(t))'}{t^a}dt+bD\int_{\beta}^{\beta+1}r(t)dt+o(1)\quad\text{as }\sigma\searrow\beta.\end{aligned}$$

By integration by parts, we have

$$\begin{aligned}C\int_{\sigma-1}^{\beta}\frac{ar(t)-(tr(t))'}{t^a}dt&=C\int_{\sigma-1}^{\beta}\frac{ar(t)}{t^a}dt-C\int_{\sigma-1}^{\beta}\frac{(tr(t))'}{t^a}dt\\&=-C(r(\beta)\beta^{1-a}-r(\sigma-1)(\sigma-1)^{1-a})\\&\qquad+C\int_{\sigma-1}^{\beta}\frac{ar(t)}{t^a}dt-C\int_{\sigma-1}^{\beta}\frac{ar(t)}{t^a}dt\\&=-C(r(\beta)\beta^{1-a}-r(\sigma-1)(\sigma-1)^{1-a}).\end{aligned}$$

By combining the above results, we have

$$\begin{aligned}&\langle R,r\rangle(s)\\&=\lim_{\sigma\searrow\beta}\langle R,r\rangle(\sigma)\\&=\lim_{\sigma\searrow\beta}\left(\sigma r(\sigma)R(\sigma)-b\int_{\sigma-1}^{\sigma}r(t+1)R(t)dt\right)\\&=\lim_{\sigma\searrow\beta}\left(\sigma r(\sigma)R(\sigma)-b\int_{\sigma-1}^{\beta}r(t+1)R(t)dt\right)\\&=\lim_{\sigma\searrow\beta}\left(\sigma r(\sigma)R(\sigma)-C(r(\beta)\beta^{1-a}-r(\sigma-1)(\sigma-1)^{1-a})+bD\int_{\beta}^{\beta+1}r(t)dt\right)\end{aligned}$$

$$= \beta r(\beta)(R(\beta) - C\beta^{-a}) + C \lim_{\sigma \searrow \beta} r(\sigma-1)(\sigma-1)^{1-a} + bD\int_{\beta}^{\beta+1} r(t)dt.$$

This completes the proof. □

10.3. **A solution of adjoint equation.** Any solution of the adjoint equation

$$(sr(s))' = ar(s) + br(s+1) \quad \text{for } s > 0,$$

can be used to study the behavior of a given solution of the original equation. Thus, it suffices to construct one particular solution for the adjoint equation for any a, b. We try to consider the Laplace transform

$$r(s) = \int_0^{\infty} \phi(x) e^{-sx} dx \quad \text{for } s > 0,$$

where for the convergence of the integral, we assume

$$\phi(x) \ll x^{-\delta} \quad (x \to 0) \quad \text{and} \quad \phi(x) \ll x^{C} \quad (x \to +\infty) \tag{10.10}$$

with some constant $\delta < 1$ and $C \geq 1$. By taking the derivative, we then have

$$r'(s) = -\int_0^{\infty} x\phi(x) e^{-sx} dx$$

By assuming the smoothness of $\phi(x)$ and using the integration by parts, we have

$$sr'(s) = \int_0^{\infty} x\phi(x)(e^{-sx})' dx = -\int_0^{\infty} (x\phi(x))' e^{-sx} dx$$

since $x\phi(x) \ll x^{1-\delta} \to 0$ as $x \searrow 0$. This gives

$$(sr(s))' = sr'(s) + r(s) = \int_0^{\infty} (-x\phi'(x)) e^{-sx} dx.$$

Thus, the requirement from the adjoint equation is given by

$$-x\phi'(x) = (a + be^{-x})\phi(x).$$

By rewriting slightly, we have

$$\frac{\phi'(x)}{\phi(x)} = -\frac{a+b}{x} + b \cdot \frac{1-e^{-x}}{x}.$$

Thus, as a candidate of $\phi(x)$, we may take

$$\phi(x) = C_0 \Phi_b(-x) x^{-(a+b)}$$

with some constant $C_0 > 0$, where

$$\Phi_b(z) := e^{b\,\mathrm{Ein}(-z)} \quad \text{for } z \in \mathbb{C}$$

and $\mathrm{Ein}(z)$ is the so called entire exponential integral defined by

$$\mathrm{Ein}(z) := \int_0^z \frac{1-e^{-t}}{t} dt \quad \text{for } z \in \mathbb{C},$$

which is obviously entire. For $x \geq 1$, we have

$$0 \leq \mathrm{Ein}(x) = \int_0^1 \frac{1-e^{-t}}{t} dt + \int_1^x \frac{dt}{t} \leq 1 + \log x \tag{10.11}$$

and so the growth condition (10.10) holds around $+\infty$ but the growth condition around 0 holds only if $a+b < 1$. We thus need to modify the contour of integration.

Before shifting of the integration, we determine the constant C_0 so that the behavior of $r(s)$ as $s \to \infty$ becomes simpler. We thus assume $a+b<1$. We consider the asymptotic behavior of

$$\int_0^\infty e^{-sx}\Phi_b(-x)x^{-(a+b)}dx$$

as $s \to \infty$. By changing the variable via $x = u/s$, we get

$$\int_0^\infty e^{-sx}\Phi_b(-x)x^{-(a+b)}dx = s^{a+b-1}\int_0^\infty e^{-u}\Phi_b\left(-\frac{u}{s}\right)u^{-(a+b)}du.$$

By Lebesgue's dominated convergence theorem, we then have

$$\begin{aligned}&\int_0^\infty e^{-sx}\Phi_b(-x)x^{-(a+b)}dx \\ &\sim s^{a+b-1}\int_0^\infty e^{-u}u^{-(a+b)}du = \Gamma(1-(a+b))s^{a+b-1} \quad (s\to\infty).\end{aligned} \tag{10.12}$$

Thus, we use the normalization

$$r(s) = \frac{1}{\Gamma(1-(a+b))}\int_0^\infty e^{-sx}\Phi_b(-x)x^{-(a+b)}dx.$$

We then shift the contour of integration. For $r \geq 0$, let us define the contours

- The straight line $\mathscr{L}_-(r)$ given by $xe^{-\pi i}$ from $x=\infty$ to $x=r$.
- The unit circle $\mathscr{C}(r)$ given by $re^{i\theta}$ from $\theta=-\pi$ to $\theta=+\pi$.
- The straight line $\mathscr{L}_+(r)$ given by $xe^{+\pi i}$ from $x=r$ to $x=\infty$.

We then define **the Hankel contour** $\mathscr{H}$ by

$$\mathscr{H}(r) := \mathscr{L}_-(r)+\mathscr{C}(r)+\mathscr{L}_+(r) \quad \text{and} \quad \mathscr{H} := \mathscr{H}(1).$$

We first assume $a+b<1$ to shift the contour.

Lemma 10.3. *For $a,b,s \in \mathbb{R}$ with $s>0$ and $a+b<1$, we have*

$$\frac{1}{2\pi i}\int_{\mathscr{H}} e^{sz}\Phi_b(z)z^{-(a+b)}dz = \frac{\sin(a+b)\pi}{\pi}\int_0^\infty e^{-sx}\Phi_b(-x)x^{-(a+b)}dx.$$

Proof. For $0<r\leq 1$, since $a+b<1$, we have

$$\left|\int_{\mathscr{C}(r)} e^{sz}\Phi_b(z)z^{-(a+b)}dz\right| \ll_{s,b} r^{1-(a+b)} \to 0 \quad \text{as } r\to 0.$$

Thus, by Cauchy's theorem, we have

$$\begin{aligned}&\frac{1}{2\pi i}\int_{\mathscr{H}} e^{sz}\Phi_b(z)z^{-(a+b)}dz \\ &= \frac{1}{2\pi i}\int_{\mathscr{L}_+(0)} e^{sz}\Phi_b(z)z^{-(a+b)}dz \\ &\qquad + \frac{1}{2\pi i}\int_{\mathscr{L}_-(0)} e^{sz}\Phi_b(z)z^{-(a+b)}dz \\ &= \frac{e^{+(1-(a+b))\pi i}}{2\pi i}\int_0^\infty e^{-sx}\Phi_b(-x)x^{-(a+b)}dx \\ &\qquad - \frac{e^{-(1-(a+b))\pi i}}{2\pi i}\int_0^\infty e^{-sx}\Phi_b(-x)x^{-(a+b)}dx\end{aligned}$$

$$= \frac{\sin(1-(a+b))\pi}{\pi} \int_0^\infty e^{-sx} \Phi_b(-x) x^{-(a+b)} dx$$
$$= \frac{\sin(a+b)\pi}{\pi} \int_0^\infty e^{-sx} \Phi_b(-x) x^{-(a+b)} dx.$$

This completes the proof. □

According to Lemma 10.3 and

$$\frac{\sin(a+b)\pi}{\pi} = \frac{1}{\Gamma(a+b)\Gamma(1-(a+b))},$$

we define **the standard solution** $r_{a,b}(s)$ by

$$r_{a,b}(s) := \frac{\Gamma(a+b)}{2\pi i} \int_{\mathscr{H}} e^{sz} \Phi_b(z) z^{-(a+b)} dz$$

for arbitrary $a, b \in \mathbb{R}$. Note that when $a+b \in \mathbb{Z}_{\leq 0}$, the pole of $\Gamma(a+b)$ is cancelled with the zero of the integral. Since the contour $\mathscr{H}$ avoids the origin $z = 0$, the above integral converges absolutely and compact uniformly with respect to $s \in (0, +\infty)$. Therefore, $r_{a,b}(s), r'_{a,b}(s)$ are analytic in any of variables $a, b, s \in \mathbb{C}$ with $\operatorname{Re} s > 0$.

The expression

$$\frac{1}{\Gamma(1-(a+b))} \int_0^\infty e^{-sx} \Phi_b(-x) x^{-(a+b)} dx$$

is analytic in a and b in the range $a+b < 1$ and coincides with the expression

$$\frac{\Gamma(a+b)}{2\pi i} \int_{\mathscr{H}} e^{sz} \Phi_b(z) z^{-(a+b)} dz$$

assuming $a+b \notin \mathbb{Z}_{\leq 0}$ by the above argument.

We check the above defined $r_{a,b}(s)$ is a solution of the adjoint equation.

Proposition 10.4. *For $a, b \in \mathbb{R}$, the standard solution $r_{a,b}(s)$ satisfies*

$$(sr_{a,b}(s))' = ar_{a,b}(s) + br_{a,b}(s+1) \quad \text{for } s > 0.$$

Proof. By the identity theorem of analytic function, we may assume $z = a+b$ is not a pole of $\Gamma(z)$. Then, recalling the definition of $\Phi_b(z)$, i.e.

$$\Phi_b(z) := e^{b \operatorname{Ein}(-z)}$$

and using the integration by parts, we have

$$\begin{aligned} sr_{a,b}(s) &= \frac{\Gamma(a+b)}{2\pi i} \int_{\mathscr{H}} (e^{sz})' e^{b \operatorname{Ein}(-z)} z^{-(a+b)} dz \\ &= -\frac{\Gamma(a+b)}{2\pi i} \int_{\mathscr{H}} e^{sz} (e^{b \operatorname{Ein}(-z)} z^{-(a+b)})' dz \\ &= b \frac{\Gamma(a+b)}{2\pi i} \int_{\mathscr{H}} e^{sz} e^{b \operatorname{Ein}(-z)} \operatorname{Ein}'(-z) z^{-(a+b)} dz \\ &\qquad + (a+b) \frac{\Gamma(a+b)}{2\pi i} \int_{\mathscr{H}} e^{sz} e^{b \operatorname{Ein}(-z)} z^{-(a+b+1)} dz. \end{aligned}$$

Since

$$\operatorname{Ein}'(-z) = \frac{1-e^{-(-z)}}{(-z)} = \frac{e^z - 1}{z},$$

we further have

$$sr_{a,b}(s) = b\frac{\Gamma(a+b)}{2\pi i}\int_{\mathscr{H}} e^{(s+1)z}e^{b\,\mathrm{Ein}(-z)}z^{-(a+b+1)}dz + a\frac{\Gamma(a+b)}{2\pi i}\int_{\mathscr{H}} e^{sz}e^{b\,\mathrm{Ein}(-z)}z^{-(a+b+1)}dz.$$

By taking the derivative with respect to s, we obtain

$$\begin{aligned}(sr_{a,b}(s))' &= b\frac{\Gamma(a+b)}{2\pi i}\int_{\mathscr{H}} e^{(s+1)z}e^{b\,\mathrm{Ein}(-z)}z^{-(a+b)}dz + a\frac{\Gamma(a+b)}{2\pi i}\int_{\mathscr{H}} e^{sz}e^{b\,\mathrm{Ein}(-z)}z^{-(a+b)}dz \\ &= ar(s) + br(s+1).\end{aligned}$$

This completes the proof. □

10.4. **Asymptotic behavior of $r_{a,b}(s)$ as $s \to \infty$.** We next study the asymptotic behavior of $r_{a,b}(s)$ as $s \to \infty$, a prototype of which is already given in (10.12).

Lemma 10.5. *For $a, b \in \mathbb{R}$ and $N \in \mathbb{Z}_{\geq 0}$ with $a+b-1 < N$, we have*

$$r_{a,b}(s) = \sum_{0 \leq n < N} \Phi_b^{(n)}(0)\binom{a+b-1}{n}s^{a+b-1-n} + \frac{1}{\Gamma(1-(a+b))}\int_0^\infty e^{-sx}R_{N,b}(-x)x^{-(a+b)}dx$$

for $s > 0$, where the binomical coefficient is defined by

$$\binom{a+b-1}{n} := \frac{(a+b-1)\cdots(a+b-n)}{n!}$$

and $R_{N,b}(x)$ is the remainder of the Taylor expansion

$$\Phi_b(z) = \sum_{0 \leq n < N} \frac{\Phi_b^{(n)}(0)}{n!}z^n + R_{N,b}(z) \tag{10.13}$$

at $z = 0$.

Proof. By the identity theorem of analytic function, we may assume $z = a+b$ is not a pole of $\Gamma(z)$. By substituting (10.13) into the definition of $r_{a,b}(s)$,

$$r_{a,b}(s) = \sum_{0 \leq n < N} \frac{\Phi_b^{(n)}(0)}{n!}\frac{\Gamma(a+b)}{2\pi i}\int_{\mathscr{H}} e^{sz}z^{n-(a+b)}dz + \frac{\Gamma(a+b)}{2\pi i}\int_{\mathscr{H}} e^{sz}R_{N,b}(z)z^{-(a+b)}dz.$$

For the former terms, by changing the variable and using Cauchy's theorem,

$$\frac{\Gamma(a+b)}{2\pi i}\int_{\mathscr{H}} e^{sz}z^{n-(a+b)}dz = \frac{\Gamma(a+b)}{2\pi i}s^{a+b-1-n}\int_{\mathscr{H}} e^{z}z^{n-(a+b)}dz.$$

By using Hankel's formula for Gamma function, we further have

$$\frac{\Gamma(a+b)}{2\pi i}\int_{\mathscr{H}} e^{sz}z^{n-(a+b)}dz = \frac{\Gamma(a+b)}{\Gamma((a+b)-n)}s^{a+b-1-n}.$$

Since the recurrence formula $\Gamma(s+1) = s\Gamma(s)$ gives

$$\frac{\Gamma(a+b)}{\Gamma((a+b)-n)} = (a+b-1)\cdots(a+b-n),$$

we obtain

$$\begin{aligned}&\sum_{0\le n<N} \frac{\Phi_b^{(n)}(0)}{n!}\frac{\Gamma(a+b)}{2\pi i}\int_{\mathscr{H}} e^{sz}z^{n-(a+b)}dz\\ &= \sum_{0\le n<N}\Phi_b^{(n)}(0)\binom{a+b-1}{n}s^{a+b-1-n}.\end{aligned}$$

For the last remainder term, since

$$|R_{N,b}(z)| \ll_{N,b} |z|^N \quad \text{for } |z|\le 1$$

and $N+1-(a+b)>0$, we have

$$\left|\int_{\mathscr{C}(r)} e^{sz}R_{N,b}(z)z^{-(a+b)}dz\right| \ll_{s,b} r^{N+1-(a+b)} \to 0 \quad \text{as } r\to 0.$$

Therefore, we can shrink the Hankel contour to obtain

$$\begin{aligned}&\frac{\Gamma(a+b)}{2\pi i}\int_{\mathscr{H}} e^{sz}R_{N,b}(z)z^{1-(a+b)}dz\\
&= \frac{\Gamma(a+b)}{2\pi i}\int_{\mathscr{L}_+(0)} e^{sz}R_{N,b}(z)z^{-(a+b)}dz\\
&\qquad + \frac{\Gamma(a+b)}{2\pi i}\int_{\mathscr{L}_-(0)} e^{sz}R_{N,b}(z)z^{-(a+b)}dz\\
&= \frac{\Gamma(a+b)}{2\pi i}e^{+(1-(a+b))\pi i}\int_0^\infty e^{-sx}R_{N,b}(-x)x^{-(a+b)}dx\\
&\qquad - \frac{\Gamma(a+b)}{2\pi i}e^{-(1-(a+b))\pi i}\int_0^\infty e^{-sx}R_{N,b}(-x)x^{-(a+b)}dx\\
&= \frac{\Gamma(a+b)}{\pi}(\sin(1-(a+b))\pi)\int_0^\infty e^{-sx}R_{N,b}(-x)x^{-(a+b)}dx\\
&= \frac{1}{\Gamma(1-(a+b))}\int_0^\infty e^{-sx}R_{N,b}(-x)x^{-(a+b)}dx.\end{aligned}$$

By combining the above results, we obtain the assertion. □

Proposition 10.6. *For $a, b\in\mathbb{R}$ and $N\in\mathbb{Z}_{\ge 0}$, we have*

$$r_{a,b}(s) = \sum_{0\le n<N}\Phi_b^{(n)}(0)\binom{a+b-1}{n}s^{a+b-1-n} + O(s^{a+b-1-N}) \tag{10.14}$$

for $s\ge 1$, where the implicit constant depends on N, a, b. In particular, we have

$$r_{a,b}(s) = s^{a+b-1} + O(s^{a+b-2}), \tag{10.15}$$

for $s\ge 1$, where the implicit constant depends on a, b, and so

$$r_{a,b}(s) \sim s^{a+b-1} \quad \text{as } s\to\infty, \tag{10.16}$$

where the rate of convergence depends on a, b.

Proof. We first prove (10.14). Let $M = M(a,b,N)$ be the smallest non-negative integer with $a+b-1 < M$ and $N \le M$. By Lemma 10.5, we have

$$r_{a,b}(s) = \sum_{0\le n<M} \Phi_b^{(n)}(0)\binom{a+b-1}{n} s^{a+b-1-n} + \frac{1}{\Gamma(1-(a+b))}\int_0^\infty e^{-sx} R_{M,b}(-x)x^{-(a+b)}dx.$$

Since

$$\sum_{N\le n<M} \Phi_b^{(n)}(0)\binom{a+b-1}{n} s^{a+b-1-n} \ll_{a,b,N} s^{a+b-1-N},$$

under the notation of Lemma 10.5, it suffices to show

$$\frac{1}{\Gamma(1-(a+b))}\int_0^\infty e^{-sx} R_{M,b}(-x)x^{-(a+b)}dx \ll s^{a+b-1-M}$$

for $s \ge 1$. We decompose the integral as

$$\int_0^\infty e^{-sx} R_{M,b}(-x)x^{-(a+b)}dx = \int_0^1 + \int_1^\infty = I_1 + I_2, \quad \text{say.}$$

In the integral I_1, we have $|-x| \le 1$. Thus, by the definition of $R_{M,b}(-x)$, we have

$$|R_{M,b}(-x)| \ll_{M,b} x^M.$$

Therefore, we have

$$\begin{aligned} I_1 &\ll \int_0^\infty e^{-sx} x^{M-(a+b)}dx \\ &\ll s^{a+b-1-M}\int_0^\infty e^{-x}x^{M-(a+b)}dx \ll s^{a+b-1-M}. \end{aligned}$$

For the integral I_2, note that

$$R_{M,b}(-x) = e^{b\,\mathrm{Ein}(x)} - \sum_{0\le n<M}\frac{\Phi_b^{(n)}(0)}{n!}x^n \ll x^{\max(b,M-1)} \le x^{b+M-1}$$

for $x \ge 1$ by (10.11). Therefore, we have

$$\begin{aligned} I_2 &\ll \int_1^\infty e^{-sx}x^{M-1-a}dx \\ &\ll e^{-\frac{s}{2}}\int_1^\infty e^{-\frac{s}{2}x}x^{M-1-a}dx \\ &\ll e^{-\frac{s}{2}}\int_s^\infty e^{-\frac{1}{2}x}x^{M-1-a}dx \ll e^{-\frac{s}{2}} \ll s^{a+b-1-M} \end{aligned}$$

since $s \ge 1$. By combining the above results, we obtain the result. □

10.5. **Special cases of $r_{a,b}(s)$.**

Lemma 10.7. *For $b \in \mathbb{R}$, we have*

$$\begin{cases} \Phi_b^{(0)}(0) = 1, \\ \Phi_b^{(n)}(0) = -\dfrac{b}{n}\displaystyle\sum_{\ell=0}^{n-1}\binom{n}{\ell}\Phi_b^{(\ell)}(0) \quad \text{for } n \ge 1. \end{cases}$$

Proof. The first formula is obvious. By taking the derivative, we have

$$\Phi_b'(z) = \left(e^{b\,\mathrm{Ein}(-z)}\right)' = -b\,\mathrm{Ein}'(-z)e^{b\,\mathrm{Ein}(-z)} = -b\left(\frac{e^z-1}{z}\right)\Phi_b(z)$$

and so

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{\Phi_b^{(n)}(0)}{(n-1)!}z^n &= -b\sum_{k=1}^{\infty}\frac{z^k}{k!}\sum_{\ell=0}^{\infty}\frac{\Phi_b^{(\ell)}(0)}{\ell!}z^\ell \\ &= -b\sum_{n=1}^{\infty}\left(\frac{1}{n!}\sum_{\ell=0}^{n-1}\binom{n}{\ell}\Phi_b^{(\ell)}(0)\right)z^n.\end{aligned}$$

By comparing the coefficients, we obtain the assertion. □

Proposition 10.8.

(i) *For $a+b=1$, we have $r_{a,b}(s)=1$.*

(ii) *We have*

$$r_{\frac{1}{2},\frac{1}{2}}(s) = 1, \quad r_{1,1}(s) = s-1, \quad r_{2,2}(s) = s^3-6s^2+9s-\tfrac{8}{3}.$$

Proof.

(i) When $a+b=1$, we can take $N=1$ in Lemma 10.5 to get

$$r_{a,b}(s) = 1 + \frac{1}{\Gamma(1-(a+b))}\int_0^\infty e^{-sx}(\Phi_b(-x)-1)x^{-(a+b)}dx$$

since $\Phi_b(0)=1$. Also, since $1-(a+b)=0$, the second term on the right-hand side is zero and so $r_{a,b}(s)=1$. This proves (i).

(ii) The first formula $r_{\frac{1}{2},\frac{1}{2}}(s)=1$ follows by (i) proven above. We use Lemma 10.5 and Lemma 10.7. By taking $N=2,4$ in Lemma 10.5, we have

$$\begin{aligned}r_{1,1}(s) &= \Phi_1^{(0)}(0)\binom{1}{0}s + \Phi_1^{(1)}(0)\binom{1}{1} \\ &= s + \Phi_1^{(1)}(0), \\ r_{2,2}(s) &= \Phi_2^{(0)}(0)\binom{3}{0}s^3 + \Phi_2^{(1)}(0)\binom{3}{1}s^2 + \Phi_2^{(2)}(0)\binom{3}{2}s + \Phi_2^{(3)}(0)\binom{3}{3} \\ &= s^3 + 3\Phi_2^{(1)}(0)s^2 + 3\Phi_2^{(2)}(0)s + \Phi_2^{(3)}(0).\end{aligned}$$

By Lemma 10.7, we have

$$\begin{aligned}\Phi_1^{(1)}(0) &= -\binom{1}{0}\Phi_1^{(0)}(0) = -1, \\ \Phi_2^{(1)}(0) &= -2\binom{1}{0}\Phi_2^{(0)}(0) = -2, \\ \Phi_2^{(2)}(0) &= -\left(\binom{2}{0}\Phi_2^{(0)}(0) + \binom{2}{1}\Phi_2^{(1)}(0)\right) = -(1-4) = 3, \\ \Phi_2^{(3)}(0) &= -\frac{2}{3}\left(\binom{3}{0}\Phi_2^{(0)}(0) + \binom{3}{1}\Phi_2^{(1)}(0) + \binom{3}{2}\Phi_2^{(2)}(0)\right) \\ &= -\frac{2}{3}(1-6+9) = -\frac{8}{3}.\end{aligned}$$

Combining the above results, we obtain the claimed formulas. □

10.6. **Asymptotic behavior of $r_{a,b}(s)$ as $s \searrow 0$.** We next study the asymptotic behavior of $r_{a,b}(s)$ as $s \searrow 0$ for some values of a, b.

Lemma 10.9. *We have*

$$\int_0^1 \frac{1-e^{-t}}{t}dt - \int_1^\infty \frac{e^{-t}}{t}dt = \gamma.$$

Proof. By integration by parts, we have

$$\begin{aligned}
&\int_0^1 \frac{1-e^{-t}}{t}dt - \int_1^\infty \frac{e^{-t}}{t}dt \\
&= \Big[(1-e^{-t})\log t\Big]_0^1 - \int_0^1 e^{-t}\log t dt - \Big[e^{-t}\log t\Big]_1^\infty - \int_1^\infty e^{-t}\log t dt \\
&= -\int_0^\infty e^{-t}\log t dt = -\Gamma'(1) = -\frac{\Gamma'(1)}{\Gamma(1)} = \gamma.
\end{aligned}$$

This proves the lemma. □

Proposition 10.10. *For $a, b \in \mathbb{R}$ with $a+b<1$ and $a<1$, we have*

$$r_{a,b}(s) \sim \frac{e^{b\gamma}\Gamma(1-a)}{\Gamma(1-(a+b))}s^{a-1} \quad \text{as } s \searrow 0.$$

Proof. Since $a+b<1$, we have

$$r_{a,b}(s) = \frac{1}{\Gamma(1-(a+b))}\int_0^\infty \exp\left(-sx + b\int_0^x \frac{1-e^{-t}}{t}dt\right)x^{-(a+b)}dx.$$

With the convention that

$$\int_1^x \frac{1-e^{-t}}{t}dt = -\int_x^1 \frac{1-e^{-t}}{t}dt \quad \text{if } x \le 1,$$

we have

$$\begin{aligned}
b\int_0^x \frac{1-e^{-t}}{t}dt &= b\int_0^1 \frac{1-e^{-t}}{t}dt + b\int_1^x \frac{1-e^{-t}}{t}dt \\
&= b\int_0^1 \frac{1-e^{-t}}{t}dt - b\int_1^x \frac{e^{-t}}{t}dt + b\log x
\end{aligned}$$

and so

$$r_{a,b}(s) = \frac{1}{\Gamma(1-(a+b))}\int_0^\infty \exp\left(-sx + b\int_0^1 \frac{1-e^{-t}}{t}dt - b\int_1^x \frac{e^{-t}}{t}dt\right)x^{-a}dx.$$

By changing variable via $sx = u$, we have

$$r_{a,b}(s) = \frac{s^{a-1}}{\Gamma(1-(a+b))}\int_0^\infty \exp\left(-x + b\int_0^1 \frac{1-e^{-t}}{t}dt - b\int_1^{\frac{x}{s}} \frac{e^{-t}}{t}dt\right)x^{-a}dx.$$

By Lebesgue's dominated convergence theorem (we need to construct the dominating function by considering two cases $x \leq s$ and $s \geq x$), as $s \searrow 0$, we have

$$\begin{aligned}\frac{r_{a,b}(s)}{s^{a-1}} &\to \frac{1}{\Gamma(1-(a+b))}\exp\left(b\int_0^1 \frac{1-e^{-t}}{t}dt - b\int_1^\infty \frac{e^{-t}}{t}dt\right)\int_0^\infty e^{-x}x^{-a}dx \\ &= \frac{\Gamma(1-a)}{\Gamma(1-(a+b))}\exp\left(b\int_0^1 \frac{1-e^{-t}}{t}dt - b\int_1^\infty \frac{e^{-t}}{t}dt\right) \\ &= \frac{e^{b\gamma}\Gamma(1-a)}{\Gamma(1-(a+b))}\end{aligned}$$

By Lemma 10.9. This completes the proof. □

10.7. **Zeros of $r_{a,b}(s)$.** Our choice of the parameter $\beta = \beta(\kappa)$ will be related to a zero of $r_{a,b}(s)$. We thus study the location of zeros of $r_{a,b}(s)$.

Proposition 10.11. *For $a, b \in \mathbb{R}$ and $s > 0$, we have*

$$r'_{a,b}(s) = (a+b-1)r_{a-1,b}(s).$$

Proof. By the identity theorem of analytic function, we may assume $z = a+b$ is not a pole of $\Gamma(z)$. By taking the derivative of

$$r_{a,b}(s) = \frac{\Gamma(a+b)}{2\pi i}\int_{\mathscr{H}} e^{sz}\Phi_b(z)z^{-(a+b)}dz,$$

we obtain

$$r'_{a,b}(s) = \frac{\Gamma(a+b)}{2\pi i}\int_{\mathscr{H}} e^{sz}\Phi_b(z)z^{-((a-1)+b)}dz.$$

If $a+b-1 \notin \mathbb{Z}_{\leq 0}$, we have

$$\begin{aligned}r'_{a,b}(s) &= \frac{\Gamma(a+b)}{\Gamma(a+b-1)}\frac{\Gamma((a-1)+b)}{2\pi i}\int_{\mathscr{H}} e^{sz}\Phi_b(z)z^{-((a-1)+b)}dz \\ &= \frac{\Gamma(a+b)}{\Gamma(a+b-1)}r_{a-1,b}(s) = (a+b-1)r_{a-1,b}(s).\end{aligned}$$

If $a+b-1 \in \mathbb{Z}_{\leq 0}$, then since $a+b \notin \mathbb{Z}_{\leq 0}$, we have $a+b=1$ and so $a+b-1<1$. Therefore, by Lemma 10.3, we have

$$\begin{aligned}r'_{a,b}(s) &= \Gamma(a+b)\left(\frac{\sin((a-1)+b)\pi}{\pi}\right)\int_0^\infty e^{sz}\Phi_b(z)z^{-((a-1)+b)}dz \\ &= 0 = (a+b-1)r_{a-1,b}(s)\end{aligned}$$

since now $\sin((a-1)+b)\pi = (a+b-1) = 0$. This completes the proof. □

Proposition 10.12. *For $a, b \in \mathbb{R}$ with $a+b<1$, we have $r_{a,b}(s) > 0$ for $s > 0$.*

Proof. Since $a+b<1$, the standard solution $r_{a,b}(s)$ is given by

$$r_{a,b}(s) = \frac{1}{\Gamma(1-(a+b))}\int_0^\infty e^{-sx}\Phi_b(-x)x^{-(a+b)}dx.$$

Since the integrand of

$$\int_0^\infty e^{-sx}\Phi_b(-x)x^{-(a+b)}dx$$

is positive and the gamma factor is non-zero, the assertion follows. □

Proposition 10.13. *For $a, b \in \mathbb{R}$, we have*
$$\#\{s \in (0, +\infty) \mid r_{a,b}(s) = 0\} < \max(a + b, 1).$$

Proof. We prove the assertion in the range
$$a + b \leq n$$
by induction on the range of n.

Initial case $n = 1$. For $n = 1$, we have $a + b \leq 1$ and so $\max(a + b, 1) = 1$. When $a+b < 1$, by Proposition 10.12, $r_{a,b}(s)$ is positive and so non-zero. When $a+b = 1$, by (i) of Proposition 10.8, we have $r_{a,b}(s) = 1$ and so $r_{a,b}(s)$ has no zero.

Induction step. Assume $n \geq 1$ and the assertion is proved for $a + b \leq n$. We shall prove the assertion for $n < a + b \leq n + 1$. Suppose that $r_{a,b}(s)$ has at least $M \geq 1$ zeros in $(0, +\infty)$. By Rolle's theorem, we can find $(M - 1)$ zeros of $r'_{a,b}(s)$ between these M zeros. However, by Proposition 10.11 and $a + b - 1 > 0$, zeros of $r'_{a,b}(s)$ coincide with zeros of $r_{a-1,b}(s)$. Since $(a - 1) + b \leq n$, by the induction hypothesis, we have $M - 1 < \max((a - 1) + b, 1)$. Therefore, by taking M to be the number of all zeros of $r_{a,b}(s)$ (or, if there is no zero, then the assertion trivially holds), we get
$$\#\{s \in (0, +\infty) \mid r_{a,b}(s)\} < \max((a - 1) + b, 1) + 1 = \max(a + b, 2).$$
If $\max(a + b, 2) = a + b$, then since $a + b \leq \max(a + b, 1)$, the assertion holds. If $\max(a + b, 2) = 2$, since the left-hand side is an integer, we have
$$\#\{s \in (0, +\infty) \mid r_{a,b}(s)\} \leq 1 < a + b = \max(a + b, 1).$$
This completes the proof. □

Proposition 10.14. *For $a, b \in \mathbb{R}$, the following are equivalent:*
(i) *We have $a + b > 1$ and $b > 0$.*
(ii) *The standard solution $r_{a,b}(s)$ has a zero in $(0, +\infty)$.*

Proof.

(i) $\Longrightarrow$ (ii). Assume $b > 0$. We prove the existence of zero in the range
$$n < a + b \leq n + 1$$
by induction on $n \in \mathbb{N}$.

Initial case $n = 1$. For the initial case $n = 1$, we have $1 < a + b \leq 2$ and so we can take $N = 1$ in Lemma 10.5. Since $\Phi_b(0) = e^{b\,\mathrm{Ein}(0)} = 1$, this gives
$$\begin{aligned} r_{a,b}(s) &= s^{a+b-1} + \frac{1}{\Gamma(1 - (a + b))} \int_0^\infty e^{-sx}(\Phi_b(-x) - 1)x^{-(a+b)}dx \\ &= s^{a+b-1}\left(1 + \frac{1}{\Gamma(1 - (a + b))} \int_0^\infty e^{-x}\left(\Phi_b\left(-\frac{x}{s}\right) - 1\right)x^{-(a+b)}dx\right). \end{aligned}$$
As $s \searrow 0$, since $b > 0$, we have
$$\int_0^\infty e^{-x}\left(\Phi_b\left(-\frac{x}{s}\right) - 1\right)x^{-(a+b)}dx = \int_0^\infty e^{-x}\left(e^{b\,\mathrm{Ein}(\frac{x}{s})} - 1\right)x^{-(a+b)}dx$$

$$\geq \int_1^\infty e^{-x}\Big(e^{b\,\mathrm{Ein}(\frac{x}{s})}-1\Big)x^{-(a+b)}dx$$
$$\geq \Big(e^{b\,\mathrm{Ein}(\frac{1}{s})}-1\Big)\int_1^\infty e^{-x}x^{-(a+b)}dx \to \infty.$$

Thus, by using

$$\frac{1}{\Gamma(1-(a+b))} = \Gamma(a+b)\frac{\sin(a+b)\pi}{\pi} < 0$$

for $1 < a+b < 2$, we obtain $r_{a,b}(s) < 0$ for sufficiently small s. Since $r_{a,b}(s) > 0$ for all large s by Proposition 10.6, the intermediate value theorem implies that $r_{a,b}(s)$ has a zero.

Induction step. We consider the induction step from the $(n-1)$-th case to the n-th case with $n \geq 2$. Assume $n < a+b \leq n+1$. By the adjoint equation

$$(sr_{a,b}(s))' = ar_{a,b}(s) + br_{a,b}(s+1),$$

we have

$$sr'_{a,b}(s) = (a-1)r_{a,b}(s) + br_{a,b}(s+1).$$

By Proposition 10.11, we have

$$(a+b-1)sr_{a-1,b}(s) = (a-1)r_{a,b}(s) + br_{a,b}(s+1).$$

By the induction hypothesis and $n-1 < (a-1)+b \leq n$, we can find the largest zero α_1 of $r_{a-1,b}(s)$. By chosing $s = \alpha_1$, we get

$$0 = (a-1)r_{a,b}(\alpha_1) + br_{a,b}(\alpha_1+1).$$

By the maximality of α_1 and Proposition 10.11, we find that $r_{a,b}(s)$ is strictly increasing for $s > \alpha_1$. Since $b > 0$, we have

$$0 = (a-1)r_{a,b}(\alpha_1) + br_{a,b}(\alpha_1+1) > (a+b-1)r_{a,b}(\alpha_1).$$

This shows $r_{a,b}(\alpha_1) < 0$. Since $r_{a,b}(s) > 0$ for all large s by Proposition 10.6, the intermediate value theorem implies that $r_{a,b}(s)$ has a zero.

(ii) $\Longrightarrow$ (i). We prove the contraposition, i.e. we prove that $r_{a,b}(s)$ has no zeros if $a+b \leq 1$ or $b \leq 0$. For the case $a+b \leq 1$, Proposition 10.13 shows that $r_{a,b}(s)$ has no zero. We thus consider the case $a+b > 1$ and $b \leq 0$. If $b = 0$, then

$$\Phi_b(x) = e^{b\,\mathrm{Ein}(x)} = 1$$

and so

$$r_{a,b}(s) = \frac{\Gamma(a+b)}{2\pi i}\int_{\mathscr{H}} e^{sz}\Phi_b(z)z^{-(a+b)}dz = \frac{\Gamma(a+b)}{2\pi i}\int_{\mathscr{H}} e^{sz}z^{-(a+b)}dz = s^{a+b-1}$$

by Hankel's formula, which shows $r_{a,b}(s)$ has no zero. Thus, we may assume $b < 0$. By the adjoint equation

$$(sr_{a,b}(s))' = ar_{a,b}(s) + br_{a,b}(s+1),$$

we have

$$sr'_{a,b}(s) = (a-1)r_{a,b}(s) + br_{a,b}(s+1).$$

Assume to the contrary that $r_{a,b}(s)$ has a zero in $(0,+\infty)$. By Proposition 10.13, we can take the largest one, say $s = \alpha$. By substituting $s = \alpha$ in the above equation,

$$\alpha r'_{a,b}(\alpha) = (a-1)r_{a,b}(\alpha) + br_{a,b}(\alpha+1) = br_{a,b}(\alpha+1).$$

By (10.16) of Proposition 10.6, we should have $r'_{a,b}(\alpha) \geq 0$ since otherwise $r_{a,b}(s) < 0$ for $s > \alpha$ sufficiently close to α and $r_{a,b}(s) > 0$ for large $s > \alpha$ and so the intermediate value theorem shows there exists a zero larger than α, which contradicts the maximality of α. Similarly, the intermediate value theorem and the maximality of α shows $r_{a,b}(\alpha + 1) > 0$. Recalling $b < 0$, we then have

$$0 \leq \alpha r'_{a,b}(\alpha) = b r_{a,b}(\alpha + 1) < 0$$

which is a contradiction. This completes the proof. □

10.8. **Propagation of the inequality.**

Lemma 10.15. *For real numbers a, b, β with $\beta \geq 1$ and functions*

$$R \in \mathsf{DDE}(a, b, \beta),$$

assume that the follwing are satisfied:

(1) *We have $b > 0$.*
(2) *For the standard solution $r(s) \coloneqq r_{a,b}(s)$, we have*

$$\langle R, r\rangle(s) = 0.$$

(3) *We have $r(s) > 0$ for $s \geq \beta$.*

When the initial conditions

$$\begin{cases} R(s) \text{ is not constantly zero on } \beta - 1 < s < \beta \\ R(s) \geq 0 \quad \text{for } \beta - 1 < s < \beta \end{cases} \tag{10.17}$$

hold, then we have:

(i) *We have $R(s) > 0$ for $s \geq \beta$.*
(ii) *The function $s^a R(s)$ is non-increasing for $s \geq \beta$.*

Proof.

(i). By the assumption (2), we have

$$sr(s)R(s) = b \int_{s-1}^{s} r(t+1)R(t)dt \quad \text{for } s > \beta. \tag{10.18}$$

By taking the limit $s \searrow \beta$, using (10.17) and recalling (1) and (3), we get

$$\beta r(\beta)R(\beta) = b \int_{\beta-1}^{\beta} r(t+1)R(t)dt > 0$$

and so $R(\beta) > 0$. Assume to the contrary to (i), suppose that $R(s) \leq 0$ for some $s \geq \beta$. By the continuity, we can then take the least $s_1 \geq \beta$ such that $R(s_1) \leq 0$. Since $R(\beta) > 0$ as we have seen, $s_1 > \beta$. Then, by the minimality of s_1, we have $R(s) \geq 0$ for $\beta - 1 < s < s_1$ and $R(s) > 0$ for s slightly smaller than s_1. Then, by (3) and (10.18), we have

$$s_1 r(s_1)R(s_1) = b \int_{s_1-1}^{s_1} r(t+1)R(t)dt > 0.$$

This contradicts the choice of s_1 and so (i) holds.

(ii). This follows by (i) proven above since

$$(s^a R(s))' = as^{a-1}R(s) + s^{a-1} \cdot sR'(s) = -bs^{a-1}R(s-1) \geq 0 \quad \text{for } s \geq \beta,$$

which follows by $R \in \mathsf{DDE}(a, b, \beta)$. □

Lemma 10.16. *For real numbers a, b, s_0 with $b, s_0 > 0$ such that*

$$r_{a,\pm b}(s) > 0 \quad \text{for } s > s_0, \tag{10.19}$$

we have

$$\left(\frac{r_{a,+b}(s)}{r_{a,-b}(s)}\right)' > 0 \quad \text{for } s > s_0.$$

Consequently, we have

$$\frac{r_{a,+b}(s)}{r_{a,-b}(s)} \text{ is strictly increasing for } s > s_0.$$

Proof. The adjoint equations

$$(sr_{a,\pm b}(s))' = ar_{a,\pm b}(s) \pm br_{a,\pm b}(s+1)$$

can be rewritten as

$$(s^{1-a}r_{a,\pm b}(s))' = s^{-a}(sr_{a,\pm b}(s))' - as^{-a}r_{a,\pm b}(s) = \pm bs^{-a}r_{a,\pm b}(s+1).$$

Therefore, by the assumption (10.19) and $b > 0$, we have

$$\begin{aligned}\left(\frac{r_{a,+b}(s)}{r_{a,-b}(s)}\right)' &= \left(\frac{s^{1-a}r_{a,+b}(s)}{s^{1-a}r_{a,-b}(s)}\right)' \\ &= \frac{(s^{1-a}r_{a,+b}(s))' \cdot s^{1-a}r_{a,-b}(s) - s^{1-a}r_{a,+b}(s) \cdot (s^{1-a}r_{a,-b}(s))'}{(s^{1-a}r_{a,-b}(s))^2} \\ &= b \cdot \frac{r_{a,+b}(s+1)r_{a,-b}(s) + r_{a,+b}(s)r_{a,-b}(s+1)}{sr_{a,-b}(s)^2} > 0 \quad \text{for } s > s_0\end{aligned}$$

and the assertion follows. □

Lemma 10.17. *For real numbers a, b, β with $b > 0, \beta \geq 1$ and functions*

$$R^{\pm} \in \mathsf{DDE}(a, \pm b, \beta),$$

assume that the follwing are satisfied:

(1) *For the standard solutions $r^{\pm}(s) := r_{a,\pm b}(s)$ of the adjoint equations*

$$(sr^{\pm}(s))' = ar^{\pm}(s) \pm br^{\pm}(s+1),$$

the Iwaniec pairing is given by

$$\langle R^{\pm}, r^{\pm}\rangle(s) = 0 \quad \text{for } s > \beta \tag{10.20}$$

(2) *We have $r^{\pm}(s) > 0$ for $s \geq \beta$.*

When the initial conditions

$$\begin{cases} R^+(s) \text{ is not constantly zero on } \beta - 1 < s < \beta \\ |R^-(s)| \leq R^+(s) \quad \text{for } \beta - 1 < s < \beta \end{cases} \tag{10.21}$$

hold, then there exists a real number $\eta = \eta(R^{\pm}) \in (0, 1)$ such that

$$|R^-(s)| < \eta R^+(s) \quad \text{for } s \geq \beta.$$

Proof. We first prove the following claim:

Claim 10.18. *For $s \geq \beta$ and $\eta \in (0, 1]$ satisfying*

$$\begin{cases} |R^-(t)| \leq \eta R^+(t) \\ R^+(t) \text{ is not constantly zero} \end{cases} \quad \text{for } s-1 < t < s.$$

We then have

$$|R^-(s)| < \eta R^+(s).$$

Proof. By (10.20), we have

$$sr^\pm(s)R^\pm(s) = \pm b \int_{s-1}^{s} r^\pm(t+1)R^\pm(t)dt \quad \text{for } s > \beta. \tag{10.22}$$

By the continuity, this equation holds even if $s = \beta$. By choosing the sign $-$, taking the absolute value and using the assumption, we have

$$\begin{aligned} sr^-(s)|R^-(s)| &\leq b \int_{s-1}^{s} r^-(t+1)|R^-(t)|dt \\ &\leq \eta b \int_{s-1}^{s} r^-(t+1)R^+(t)dt. \end{aligned}$$

Since $R^+(t)$ is not constantly zero on $(s-1, s)$, by Lemma 10.16 and (2), we have

$$\begin{aligned} sr^-(s)|R^-(s)| &\leq \eta b \int_{s-1}^{s} \frac{r^-(t+1)}{r^+(t+1)} r^+(t+1)R^+(t)dt \\ &< \frac{r^-(s)}{r^+(s)} \cdot \eta b \int_{s-1}^{s} r^+(t+1)R^+(t)dt. \end{aligned}$$

By using (10.22) with the sign $+$, we get

$$sr^-(s)|R^-(s)| < \frac{r^-(s)}{r^+(s)} \eta s r^+(s)R^+(s) = \eta s r^-(s)R^+(s)$$

and so $|R^-(s)| < \eta R^+(s)$. This proves the claim. □

We first prove $|R^-(s)| < R^+(s)$ for $s \geq \beta$. Assume the contrary. Then, by the continuity of $R^\pm(s)$ for $s \geq \beta$, we can take the smallest $s_1 \geq \beta$ such that $|R^-(s_1)| \geq R^+(s_1)$. By the minimality of s_1 and (10.21), we have $|R^-(t)| \leq R^+(t)$ for $s_1 - 1 < t < s_1$. If $s_1 = \beta$, by the assumption, $R^+(t)$ is not constantly zero for $s_1 - 1 < t < s_1$. If $s_1 > \beta$, by the minimality of s_1, we have $|R^-(t)| < R^+(t)$ with $t \geq \beta$ slightly smaller than s_1 and so again $R^+(t)$ is not constantly zero for $s_1 - 1 < t < s_1$. We can thus apply Claim to get $|R^-(s_1)| < R^+(s_1)$, a contradiction. Thus, $|R^-(s)| < R^+(s)$ for $s \geq \beta$ and so $R^+(s) \neq 0$ for $s \geq \beta$.

Since $|R^-(s)| < R^+(s)$ for $s \geq \beta$ and since $R^\pm(s)$ are continuous for $s \geq \beta$, we can take $\eta \in (0, 1)$ such that $|R^-(s)| \leq \eta R^+(s)$ for $\beta \leq s \leq \beta + 1$. We then prove $|R^-(s)| \leq \eta R^+(s)$ for $s \geq \beta$. Assume the contrary. Then, by the continuity of $R^\pm(s)$ for $s \geq \beta$, we can take the smallest $s_1 \geq \beta+1$ such that $|R^-(s_1)| \geq \eta R^+(s_1)$. By the minimality of s_1 and by the choice of η, we have $|R^-(t)| \leq \eta R^+(t)$ for $s_1 - 1 < t < s_1$. Since $R^+(s) \neq 0$ for $s \geq \beta$ as proved in the previous paragraph, we can apply the claim to get $|R^-(s_1)| < \eta R^+(s_1)$, a contradiction. This completes the proof. □

10.9. **Some integral inequalities.** To study the asymptotic behavior of the solution $R(s)$ of the original delay-differential equation, we need some integral inequalities. In this subsection, we develop such integral inequalities.

We start with introducing a function $\xi(u)$ following, e.g. Hildebrand and Tenenbaum [3, Section 2]

Proposition 10.19. *The function*

$$\eta(\xi) := \frac{e^{\xi} - 1}{\xi} = \int_0^1 e^{t\xi} dt$$

is increasing for all $\xi \in \mathbb{R}$ and

$$\lim_{\xi \to -\infty} \eta(\xi) = 0, \quad \eta(0) = 1, \quad \lim_{\xi \to +\infty} \eta(\xi) = +\infty.$$

Proof. This is obvious from the expression

$$\eta(\xi) = \int_0^1 e^{t\xi} dt.$$

This completes the proof. □

By Proposition 10.19, we can define an increasing function

$$\xi \colon (0, +\infty) \to \mathbb{R}$$

as the inverse function of

$$\eta \colon \mathbb{R} \to (0, +\infty),$$

i.e. we define $\xi(s)$ be the unique real number $\xi = \xi(s)$ such that

$$\frac{e^{\xi} - 1}{\xi} = s.$$

We then prepare some basic facts on the function $\xi(s)$.

Proposition 10.20.

(i) *We have*

$$\xi(s) = \log s + \log\log s + \frac{\log\log s}{\log s} + O\biggl(\biggl(\frac{\log\log s}{\log s}\biggr)^2\biggr)$$

for $s \geq e^e$.

(ii) *We have*

$$s - \frac{s-1}{\xi(s)} > \frac{1}{2}$$

and

$$\xi'(s) = \frac{1}{s - \frac{s-1}{\xi(s)}}$$

for $s > 1$. Consequently, we have

$$\xi'(s) = \frac{1}{s} \exp\biggl(O\biggl(\frac{1}{\log s}\biggr)\biggr)$$

for $s \geq e$.

(iii) *We have*

$$\xi''(s) = -\frac{1 - \frac{1}{\xi(s)} + \frac{\xi'(s)(s-1)}{\xi(s)^2}}{(s - \frac{s-1}{\xi(s)})^2}$$

for $s > 1$. *Consequently, we have*

$$\xi''(s) = -\frac{1}{s^2}\exp\left(O\left(\frac{1}{\log s}\right)\right)$$

for $s \geq e$.

Proof.

(i) We may assume s is sufficiently large. By the defining equation

$$\frac{e^{\xi(s)} - 1}{\xi(s)} = s,$$

we have

$$\xi(s) - \log\xi(s) + O(e^{-\xi(s)}) = \log s. \tag{10.23}$$

Since $\xi(s) \to \infty$ as $s \to \infty$, this gives

$$\xi(s) \asymp \log s.$$

On re-inserting this formula into (10.23), we get

$$\xi(s) = \log s + \log\log s + O(1).$$

On re-inserting this formula into (10.23), we further get

$$\begin{aligned}\xi(s) &= \log s + \log\xi(s) + O\left(\frac{1}{s\log s}\right)\\ &= \log s + \log\log s + \log\left(1 + \frac{\log\log s}{\log s}\right) + O\left(\frac{1}{\log s}\right)\\ &= \log s + \log\log s + \frac{\log\log s}{\log s} + O\left(\frac{1}{\log s}\right).\end{aligned}$$

A further iteration gives

$$\begin{aligned}\xi(s) &= \log s + \log\xi(s) + O\left(\frac{1}{s\log s}\right)\\ &= \log s + \log\log s + \log\left(1 + \frac{\log\log s}{\log s}\right) + O\left(\frac{\log\log s}{(\log s)^2}\right)\\ &= \log s + \log\log s + \frac{\log\log s}{\log s} + O\left(\left(\frac{\log\log s}{\log s}\right)^2\right).\end{aligned}$$

This completes the proof.

(ii) By the formula for the derivative of inverse function, we have

$$\xi'(s) = \frac{1}{\eta'(\xi(s))} \quad \text{for } s > 0. \tag{10.24}$$

Note that

$$\eta'(\xi) = \int_0^1 te^{t\xi}dt > \frac{1}{2} \quad \text{for } \xi > 0. \tag{10.25}$$

For $\xi > 0$, by integration by parts, we have

$$\eta'(\xi) = \frac{e^\xi}{\xi} - \frac{1}{\xi}\int_0^1 e^{t\xi}dt = \frac{e^\xi}{\xi} - \frac{\eta(\xi)}{\xi}$$
$$= \frac{e^\xi - 1}{\xi} - \frac{\eta(\xi)-1}{\xi} = \eta(\xi) - \frac{\eta(\xi)-1}{\xi}.$$

For $s > 1$, by substituting $\xi = \xi(s) > \xi(1) = 0$, we get

$$\eta'(\xi) = s - \frac{s-1}{\xi(s)}.$$

By (10.24) and (10.25), we get the first half of the assertion. The latter asymptotic formula is then a corollary of the former one since $\eta'(\xi(s)) \geq \frac{1}{2}$ for $s \geq 1$.

(iii) The first equation is a corollary of (ii). The latter asymptotic formula also follows from (ii) for large s. For small $s \geq e$, say $e \leq s \leq s_0$, it suffices to see $\xi''(s) \asymp 1$. By the first half of (iii), we have

$$\xi''(s) \geq \left(1 - \frac{1}{\xi(e)}\right)(\xi'(s))^2 \gg 1$$

since

$$\eta(1) = \int_0^1 e^t dt = e - 1 < e \quad \text{and so} \quad 1 < \xi(e).$$

We also have $\xi''(s) \ll 1$ by the continuity of $\xi''(s)$. This completes the proof. □

For the decay of the special solutions, we use the following lemma.

Lemma 10.21. *Let $\kappa > 0$ and $E \geq 1$. Then, for any continuous function $f(s)$ on $[s_0, +\infty)$ with $s_0 \geq 0$ satisfying the inequality*

$$(s - E)|f(s)| \leq \kappa \int_{s-1}^{s} |f(t)|dt \quad \text{for large } s \geq s_0 + 1 \tag{10.26}$$

obeys the bound

$$|f(s)| < \exp\left(-\int_\kappa^s \xi\left(\frac{t}{\kappa}\right)dt + c\log(s+e)\right)$$

for $s \geq s_0$ with some constant $c = c(f) \geq 1$.

Proof. We first prove a preliminary inequality. Take $c \geq 1$ chosen later. In this proof, implicit constants may depend on κ, E but should be independent of c. Let

$$\phi(s) := \int_\kappa^s \xi\left(\frac{t}{\kappa}\right)dt - c\log s$$

for $s \geq \kappa$. By taking the derivatives, we have

$$\phi'(s) = \xi\left(\frac{s}{\kappa}\right) - \frac{c}{s} \quad \text{and} \quad \phi''(s) = \frac{1}{\kappa}\xi'\left(\frac{s}{\kappa}\right) + \frac{c}{s^2} \tag{10.27}$$

and so by Proposition 10.20, we have

$$\left.\begin{aligned} \phi'(s) &= \xi\left(\frac{s}{\kappa}\right)\left(1+O\left(\frac{c}{s\log s}\right)\right) \\ &= \xi\left(\frac{s}{\kappa}\right)\left(1+O\left(\frac{1}{s}\right)\right) \\ \phi''(s) &\ll \frac{1}{s} \end{aligned}\right\} \quad \text{for } s \geq s_1(c,\kappa) \tag{10.28}$$

By the mean value theorem, we then have

$$\begin{aligned} \phi'(t) &= \phi'(s) + O\left(\frac{1}{s}\right) \\ &= \xi\left(\frac{s}{\kappa}\right)e^{O(\frac{1}{s})} \quad \text{for } s-1 \leq t \leq s \text{ with } s \geq s_1(c,\kappa). \end{aligned} \tag{10.29}$$

By (10.26), we have

$$\begin{aligned} |f(s)|e^{\phi(s)} &\leq \frac{\kappa e^{\phi(s)}}{s-E}\int_{s-1}^{s}|f(t)|e^{\phi(t)}\cdot e^{-\phi(t)}dt \\ &\leq \frac{\kappa e^{\phi(s)}}{s-E}\left(\int_{s-1}^{s}e^{-\phi(t)}dt\right)\max_{s-1\leq t\leq s}|f(t)|e^{\phi(t)} \end{aligned} \tag{10.30}$$

for large $s \geq s_0+1$. For the last integral, by (10.29), we have

$$\begin{aligned} e^{\phi(s)}\int_{s-1}^{s}e^{-\phi(t)}dt &= \frac{e^{\phi(s)}}{\xi(\frac{s}{\kappa})}\left(\int_{s-1}^{s}e^{-\phi(t)}\phi'(t)dt\right)e^{O(\frac{1}{s})} \\ &= \frac{e^{\phi(s)-\phi(s-1)}-1}{\xi(\frac{s}{\kappa})}e^{O(\frac{1}{s})} \quad \text{for } s \geq s_1(c,\kappa). \end{aligned}$$

By Taylor approximation, (10.27) and (10.28), for some $\sigma \in (s-1,s)$, we have

$$\begin{aligned} \phi(s)-\phi(s-1) &= \phi'(s) + \frac{1}{2}\phi''(\sigma) \\ &= \xi\left(\frac{s}{\kappa}\right) - \frac{c}{s} + O\left(\frac{1}{s}\right) \quad \text{for } s \geq s_1(c,\kappa). \end{aligned}$$

Therefore, since

$$e^{\xi(\frac{s}{\kappa})} \geq e^{\xi(\frac{s}{\kappa})-\frac{c}{s}} \gg s\log s \quad \text{for } s \geq s_1(c,\kappa),$$

by using the definition of $\xi(\frac{s}{\kappa})$, we have

$$\begin{aligned} e^{\phi(s)}\int_{s-1}^{s}e^{-\phi(t)}dt &= \frac{e^{\xi(\frac{s}{\kappa})-\frac{c}{s}}}{\xi(\frac{s}{\kappa})}e^{O(\frac{1}{s})} \\ &= \frac{e^{\xi(\frac{s}{\kappa})}-1}{\xi(\frac{s}{\kappa})}e^{-\frac{c}{s}+O(\frac{1}{s})} \\ &= \frac{s}{\kappa}e^{-\frac{c}{s}+O(\frac{1}{s})} \quad \text{for } s \geq s_1(c,\kappa). \end{aligned}$$

On inserting this estimate into (10.30), we have

$$|f(s)|e^{\phi(s)} \leq \frac{1}{1-\frac{E}{s}}e^{-\frac{c}{s}+O(\frac{1}{s})}\max_{s-1\leq t\leq s}|f(t)|e^{\phi(t)}$$

$$\le e^{-\frac{c}{s}+\frac{C}{s}} \max_{s-1\le t\le s} |f(t)|e^{\phi(t)} \quad \text{for } s \ge s_1(c,\kappa,E,s_0).$$

with some $C = C(E,\kappa) \ge 0$. By taking

$$c \coloneqq C + 1,$$

we arrive at

$$|f(s)|e^{\phi(s)} \le e^{-\frac{1}{s}} \max_{s-1\le t\le s} |f(t)|e^{\phi(t)} \quad \text{for } s \ge s_1(\kappa,E,s_0). \tag{10.31}$$

We refer the $s_1(\kappa,E,s_0)$ in (10.31) as s_1 below.

We now prove the assertion. Since $\log(s+e) \asymp \log s$ for $s \ge e$, it suffices to show

$$|f(s)| < L\exp\left(-\int_\kappa^s \xi\left(\frac{t}{\kappa}\right)dt + c\log s\right) \quad \text{for } s \ge s_1, \tag{10.32}$$

where

$$L \coloneqq \max_{s_1-1\le s\le s_1} |f(s)|e^{\phi(s)}.$$

Indeed, for the original assertion with $s \ge s_0$, we can replace c by some larger value. Assume that (10.32) does not hold. Then, there should be some $s \ge s_1$ with

$$|f(s)|e^{\phi(s)} \ge L.$$

By the continuity of $|f(s)|e^{\phi(s)}$, we can take smallest $\widetilde{s} \ge s_1$ with

$$|f(\widetilde{s})|e^{\phi(\widetilde{s})} \ge L.$$

By the minimality of $\widetilde{s}$ and the definition of L, we have

$$|f(s)|e^{\phi(s)} < L \quad \text{for } s_1 - 1 \le s \le \widetilde{s}.$$

By (10.31), We then have

$$L \le |f(\widetilde{s})|e^{\phi(\widetilde{s})} \le e^{-\frac{1}{\widetilde{s}}} \max_{\widetilde{s}-1\le t\le \widetilde{s}} |f(t)|e^{\phi(t)} \le e^{-\frac{1}{\widetilde{s}}} L,$$

which is a contradiction. Therefore, the assertion must be true. □

Later, we need the following lower-bound version of Lemma 10.21.

Lemma 10.22. *Let $\kappa > 0$ and $E \ge 1$. Then, for any positive continuous function $f(s)$ on $[s_0, +\infty)$ with $s_0 \ge 0$ satisfying the inequality*

$$(s+E)f(s) \ge \kappa \int_{s-1}^s f(t)dt \quad \text{for large } s \ge s_0 + 1 \tag{10.33}$$

obeys the bound

$$f(s) > \exp\left(-\int_\kappa^s \xi\left(\frac{t}{\kappa}\right)dt - c\log(s+e)\right)$$

for $s \ge s_0$ with some constant $c = c(f) \ge 1$.

Proof. We first prove a preliminary inequality. Take $c \ge 1$ chosen later. In this proof, implicit constants may depend on κ, E but should be independent of c. Let

$$\phi(s) \coloneqq \int_\kappa^s \xi\left(\frac{t}{\kappa}\right)dt + c\log s$$

for $s \geq \kappa$. By taking the derivatives, we have

$$\phi'(s) = \xi\left(\frac{s}{\kappa}\right) + \frac{c}{s} \quad \text{and} \quad \phi''(s) = \frac{1}{\kappa}\xi'\left(\frac{s}{\kappa}\right) - \frac{c}{s^2} \tag{10.34}$$

and so by Proposition 10.20, we have

$$\left.\begin{aligned} \phi'(s) &= \xi\left(\frac{s}{\kappa}\right)\left(1 + O\left(\frac{c}{s\log s}\right)\right) \\ &= \xi\left(\frac{s}{\kappa}\right)\left(1 + O\left(\frac{1}{s}\right)\right) \\ \phi''(s) &\ll \frac{1}{s} \end{aligned}\right\} \quad \text{for } s \geq s_1(c,\kappa) \tag{10.35}$$

By the mean value theorem, we then have

$$\begin{aligned} \phi'(t) &= \phi'(s) + O\left(\frac{1}{s}\right) \\ &= \xi\left(\frac{s}{\kappa}\right)e^{O(\frac{1}{s})} \quad \text{for } s-1 \leq t \leq s \text{ with } s \geq s_1(c,\kappa). \end{aligned} \tag{10.36}$$

By (10.33), we have

$$\begin{aligned} f(s)e^{\phi(s)} &\geq \frac{\kappa e^{\phi(s)}}{s+E}\int_{s-1}^{s} f(t)e^{\phi(t)} \cdot e^{-\phi(t)}dt \\ &\geq \frac{\kappa e^{\phi(s)}}{s+E}\left(\int_{s-1}^{s} e^{-\phi(t)}dt\right)\min_{s-1\leq t\leq s} f(t)e^{\phi(t)} \end{aligned} \tag{10.37}$$

for $s \geq s_0 + 1$. For the last integral, by (10.36), we have

$$\begin{aligned} e^{\phi(s)}\int_{s-1}^{s} e^{-\phi(t)}dt &= \frac{e^{\phi(s)}}{\xi(\frac{s}{\kappa})}\left(\int_{s-1}^{s} e^{-\phi(t)}\phi'(t)dt\right)e^{O(\frac{1}{s})} \\ &= \frac{e^{\phi(s)-\phi(s-1)} - 1}{\xi(\frac{s}{\kappa})}e^{O(\frac{1}{s})} \quad \text{for } s \geq s_1(c,\kappa). \end{aligned}$$

By Taylor approximation, (10.34) and (10.35), for some $\sigma \in (s-1, s)$, we have

$$\begin{aligned} \phi(s) - \phi(s-1) &= \phi'(s) + \frac{1}{2}\phi''(\sigma) \\ &= \xi\left(\frac{s}{\kappa}\right) + \frac{c}{s} + O\left(\frac{1}{s}\right) \quad \text{for } s \geq s_1(c,\kappa). \end{aligned}$$

Therefore, since

$$e^{\xi(\frac{s}{\kappa})+\frac{c}{s}} \geq e^{\xi(\frac{s}{\kappa})} \gg s\log s \quad \text{for } s \geq s_1(c,\kappa),$$

by using the definition of $\xi(\frac{s}{\kappa})$, we have

$$\begin{aligned} e^{\phi(s)}\int_{s-1}^{s} e^{-\phi(t)}dt &= \frac{e^{\xi(\frac{s}{\kappa})+\frac{c}{s}}}{\xi(\frac{s}{\kappa})}e^{O(\frac{1}{s})} \\ &= \frac{e^{\xi(\frac{s}{\kappa})} - 1}{\xi(\frac{s}{\kappa})}e^{\frac{c}{s}+O(\frac{1}{s})} \\ &= \frac{s}{\kappa}e^{\frac{c}{s}+O(\frac{1}{s})} \quad \text{for } s \geq s_1(c,\kappa). \end{aligned}$$

On inserting this estimate into (10.37), we have

$$f(s)e^{\phi(s)} \geq \frac{1}{1+\frac{E}{s}} e^{\frac{c}{s}+O(\frac{1}{s})} \min_{s-1\leq t\leq s} f(t)e^{\phi(t)}$$
$$\geq e^{\frac{c}{s}-\frac{C}{s}} \min_{s-1\leq t\leq s} f(t)e^{\phi(t)} \quad \text{for } s \geq s_1(c,\kappa,E,s_0).$$

with some $C = C(E,\kappa) \geq 0$. By taking

$$c \coloneqq C + 1,$$

we arrive at

$$f(s)e^{\phi(s)} \geq e^{\frac{1}{s}} \min_{s-1\leq t\leq s} f(t)e^{\phi(t)} \quad \text{for } s \geq s_1(\kappa,E,s_0). \tag{10.38}$$

We refer the $s_1(\kappa,E,s_0)$ in (10.38) as s_1 below.

We now prove the assertion. Since $\log(s+e) \asymp \log s$ for $s \geq e$, it suffices to show

$$f(s) > L\exp\left(-\int_\kappa^s \xi\left(\frac{t}{\kappa}\right)dt - c\log s\right) \quad \text{for } s \geq s_1, \tag{10.39}$$

where

$$L \coloneqq \min_{s_1-1\leq s\leq s_1} f(s)e^{\phi(s)}.$$

Indeed, for the original assertion with $s \geq s_0$, we can replace c by some larger value. Assume that (10.39) does not hold. Then, there should be some $s \geq s_1$ with

$$f(s)e^{\phi(s)} \leq L.$$

By the continuity of $f(s)e^{\phi(s)}$, we can take smallest $\widetilde{s} \geq s_1$ with

$$f(\widetilde{s})e^{\phi(\widetilde{s})} \leq L.$$

By the minimality of $\widetilde{s}$ and the definition of L, we have

$$f(s)e^{\phi(s)} > L \quad \text{for } s_1 - 1 \leq s \leq \widetilde{s}.$$

By (10.38), We then have

$$L \geq f(\widetilde{s})e^{\phi(\widetilde{s})} \geq e^{\frac{1}{\widetilde{s}}} \min_{\widetilde{s}-1\leq t\leq\widetilde{s}} f(t)e^{\phi(t)} \geq e^{\frac{1}{\widetilde{s}}}L,$$

which is a contradiction. Therefore, the assertion must be true. □

It is useful to prove an asymptotic estimate for the integral

$$\int_\kappa^s \xi\left(\frac{t}{\kappa}\right)dt$$

as given in the next proposition.

Proposition 10.23. *For $\kappa > 0$ and $s \geq \max(\kappa, e^e)$, we have*

$$\int_\kappa^s \xi\left(\frac{t}{\kappa}\right)dt = s\log s + s\log\log s - s\log e\kappa + \frac{s\log\log s}{\log s} + O\left(\frac{s}{\log s}\right),$$

where the implcit constant depends only on κ. Consequently, we have

$$\int_\kappa^s \xi\left(\frac{t}{\kappa}\right)dt \geq s\log s + s\log\log s - s\log e\kappa$$

for large $s \geq s_0(\kappa)$.

Proof. By changing variable via $t=\kappa u$, we have

$$\int_{\kappa}^{s}\xi\left(\frac{t}{\kappa}\right)dt=\kappa\int_{1}^{\frac{s}{\kappa}}\xi(u)du=\kappa\int_{e^e}^{\frac{s}{\kappa}}\xi(u)du+O(1).$$

By using (i) of Proposition 10.20, we have

$$\int_{\kappa}^{s}\xi\left(\frac{t}{\kappa}\right)dt=\kappa\int_{e^e}^{\frac{s}{\kappa}}\left(\log u+\log\log u+\frac{\log\log u}{\log u}\right)du$$
$$+O\left(1+\kappa\int_{e^e}^{\frac{s}{\kappa}}\left(\frac{\log\log u}{\log u}\right)^2du\right).$$

Then, by using the formulas

$$\begin{aligned}
\kappa\int_{e^e}^{\frac{s}{\kappa}}\log u du&=\kappa\left(\frac{s}{\kappa}\log\frac{s}{\kappa}-\frac{s}{\kappa}\right)+O(1)\\
&=s\log s-s\log e\kappa+O(1),\\
\kappa\int_{e^e}^{\frac{s}{\kappa}}\log\log u du&=\kappa\Big[u\log\log u\Big]_{e^e}^{\frac{s}{\kappa}}-\kappa\int_{e^e}^{\frac{s}{\kappa}}\frac{du}{\log u}\\
&=s\log\log\frac{s}{\kappa}+O\left(\frac{s}{\log s}\right)\\
&=s\log\log s+O\left(\frac{s}{\log s}\right),\\
\kappa\int_{e^e}^{\frac{s}{\kappa}}\frac{\log\log u}{\log u}du&=\frac{s\log\log s}{\log s}+O\left(\frac{s\log\log s}{(\log s)^2}+\int_{e^e}^{\frac{s}{\kappa}}\frac{\log\log u}{(\log u)^2}du\right)\\
&=\frac{s\log\log s}{\log s}++O\left(\frac{s\log\log s}{(\log s)^2}\right)
\end{aligned}$$

and

$$\int_{e^e}^{\frac{s}{\kappa}}\left(\frac{\log\log u}{\log u}\right)^2du=\int_{e^e}^{e^e\sqrt{s}}+\int_{e^e\sqrt{s}}^{\frac{s}{\kappa}}\ll\sqrt{s}+\frac{s(\log\log s)^2}{(\log s)^2}\ll\frac{s(\log\log s)^2}{(\log s)^2},$$

we obtain the assertion. □

10.10. **Dichotomy in the decay of solutions.** We now consider the decay of the solutions $R(s)$ of the original delay-differential equation with the aid of the standard solutions $r_{a,b}(s)$. As a result, we prove that one particular solution can be distinguished from the others by checking the decay as $s\to\infty$.

Lemma 10.24. *For $R\in\mathsf{DDE}(a,b,\beta)$ with $\beta\geq 1$, if*

$$\langle R,r_{a,b}\rangle(s)=0$$

then, we have

$$R(s)\ll\exp(-s\log s-s\log\log s+s\log e|b|)$$

for $s>\max(e^e,\beta)$, where the implicit constant depends on R,a,b.

Proof. Note that if $b = 0$, then $\langle R, r_{a,b}\rangle(s) = 0$ gives $sr_{a,b}(s)R(s) = 0$ and so $R(s) = 0$ for $s > \beta$. By the assumption $\langle R, r\rangle(\beta+1) = 0$ and Lemma 10.1, we have

$$sr_{a,b}(s)R(s) = b\int_{s-1}^{s} r_{a,b}(t+1)R(t)dt.$$

By the asymptotic formula

$$r_{a,b}(s) = s^{a+b-1}\left(1 + O\left(\frac{1}{s}\right)\right)$$

given as (10.15) of Proposition 10.6, we have

$$s^{a+b}\left(1 + O\left(\frac{1}{s}\right)\right)R(s) = b\int_{s-1}^{s} t^{a+b-1}\left(1 + O\left(\frac{1}{t}\right)\right)R(t)dt.$$

By taking the absolute value, we have

$$s^{a+b}\left(1 + O\left(\frac{1}{s}\right)\right)|R(s)| \le s^{a+b-1}\left(1 + O\left(\frac{1}{s}\right)\right)|b|\int_{s-1}^{s} |R(t)|dt$$

and so there is a constant E depending on a, b such that

$$(s - E)|R(s)| \le |b|\int_{s-1}^{s} |R(t)|dt \quad \text{for large } s.$$

Then, the assertion follows by Lemma 10.21 and the latter half of Proposition 10.23.

□

Lemma 10.25. *For real numbers a, b, β with $\beta \ge 1$ and functions*

$$R \in \mathsf{DDE}(a, b, \beta),$$

assume that the follwing are satisfied:

(1) *We have $b > 0$.*
(2) *For the standard solution $r(s) := r_{a,b}(s)$, we have*

$$\langle R, r\rangle(s) = 0.$$

(3) *We have $r(s) > 0$ for $s \ge \beta$.*

When the initial conditions

$$\begin{cases} R(s) \text{ is not constantly zero on } \beta - 1 < s < \beta \\ R(s) \ge 0 \quad \text{for } \beta - 1 < s < \beta \end{cases}$$

Then, we have

$$R(s) = \exp\left(-s\log s - s\log\log s + s\log eb + O\left(\frac{s\log\log 3s}{\log 2s}\right)\right)$$

for $s \ge \beta$, where the implicit constant depends on R.

Proof. By Lemma 10.15, we know that $R(s) > 0$ for $s \ge \beta$. By the continuity, we may assume s is sufficiently large. By (2), we have

$$sr(s)R(s) = b\int_{s-1}^{s} r(t+1)R(t)dt \quad \text{for } s > \beta.$$

By Proposition 10.6 and the positivity of $R(s)$, we have

$$(s + O(1))R(s) = b\int_{s-1}^{s} R(t)dt \quad \text{for } s > \beta.$$

Therefore, by Lemma 10.21 and Lemma 10.22, we have

$$R(s)=\exp\left(-\int_{\kappa}^{s}\xi\left(\frac{t}{\kappa}\right)dt+O(\log s)\right).$$

Then, the assertion follows by Proposition 10.23. □

Lemma 10.26. *For $R\in \mathsf{DDE}(a,b,\beta)$ with $\beta\geq 1$, if*

$$\langle R,r\rangle(s)=C$$

then, we have

$$R(s)=s^{-(a+b)}\left(C+O\left(\frac{1}{s}\right)\right)$$

for $s>\beta$ where the implicit constant depends on R,a,b.

Proof. We prepare a preliminary estimate. By the assumption, we have

$$sr_{a,b}(s)R(s)=b\int_{s-1}^{s}r_{a,b}(t+1)R(t)dt+C\quad\text{for } s>\beta.$$

By the asymptotic formula

$$r_{a,b}(s)=s^{a+b-1}\left(1+O\left(\frac{1}{s}\right)\right)$$

given as (10.15) of Proposition 10.6, we have

$$sR(s)=b\int_{s-1}^{s}\frac{r_{a,b}(t+1)}{r_{a,b}(s)}R(t)dt+Cs^{-(a+b-1)}\left(1+O\left(\frac{1}{s}\right)\right)$$

and so

$$s^{a+b}R(s)-C=b\int_{s-1}^{s}\frac{r_{a,b}(t+1)}{r_{a,b}(s)}s^{a+b-1}R(t)dt+O\left(\frac{1}{s}\right)$$

By taking the absolute value, we have

$$\begin{aligned}|s^{a+b}R(s)-C|&\ll|b|\int_{s-1}^{s}t^{a+b}|R(t)|\frac{dt}{t}+\frac{1}{s}\\&\ll\int_{s-1}^{s}|t^{a+b}R(t)-C|\frac{dt}{t}+\frac{1}{s}\end{aligned}$$

and so

$$s|s^{a+b}R(s)-C|\leq L\left(s\int_{s-1}^{s}t|t^{a+b}R(t)-C|\frac{dt}{t^2}+1\right).\tag{10.40}$$

for $s\geq\beta+1$ with some constant $L\geq 1$. We now prove the assertion. It suffices to consider the range $s\geq 4L$. Assume that the assertion does not hold. Then, we have some $s\geq 4L$ such that

$$s|s^{a+b}R(s)-C|\geq AL\quad\text{with}\quad A:=\max\left(4,\max_{\beta\leq s\leq 4L}s|s^{a+b}R(s)-C|\right)\geq 1.$$

Take the smallest such $s\geq 4L$, say $s=s_1$. Then, by the minimality of s_1,

$$s|s^{a+b}R(s)-C|\leq AL\quad\text{for }\beta\leq s\leq s_1.$$

Thus, by (10.40), we have

$$AL \leq s_1|s_1^{a+b}R(s_1) - C| \leq AL^2 s_1 \int_{s_1-1}^{s_1} \frac{dt}{t^2} + L = \frac{1}{s_1 - 1}AL^2 + L$$

and since $s \geq 4L$, we have

$$AL \leq \frac{1}{2L}AL^2 + L = \frac{1}{2}AL + L \leq \frac{3}{4}AL,$$

which is a contradiction. This completes the proof. □

10.11. **Local behavior of solutions.** In Lemma 10.26, we estimated the decay of $R \in \mathsf{DDE}(a, b\beta)$ with vanishing Iwaniec paring.

Lemma 10.27. *For real numbers a, b and $\beta \geq 1$ and functions*

$$R \in \mathsf{DDE}(a, b, \beta),$$

assume that the following holds:

(i) *We have $a, b > 0$.*

(ii) *We have*

$$\begin{cases} R(s) \text{ is not constantly zero} \\ R(s) \geq 0 \end{cases} \quad \text{for } \beta - 1 < s < \beta$$

(iii) *The Iwaniec pairing vanishes, i.e. $\langle R, r\rangle(s) = 0$ with $r(s) := r_{a,b}(s)$.*

(iv) *We have $r(s) > 0$ for $s \geq \beta$.*

We then have

$$sR(s) \leq bR(s-1)\left(1 + \left(\frac{1}{s}\right)\right) \quad \text{for } s \geq \beta + 1,$$

where the implicit constant depends only on R.

Proof. By (i), (ii), (iii) and (iv), we can apply Lemma 10.15 to $R(s)$ and conclude that $R(s)$ is positive, continuous and decreasing for $s \geq \beta$. Thus, it suffices to consider large $s \geq s_0(R) \geq \beta$. Since $\langle R, r\rangle(s) = 0$,

$$sr(s)R(s) = b\int_{s-1}^{s} r(t+1)R(t)dt \quad \text{for } s \geq s_0.$$

By Proposition 10.6, this gives

$$sR(s) = b\left(\int_{s-1}^{s} R(t)dt\right)\left(1 + \left(\frac{1}{s}\right)\right).$$

Since $R(s)$ is decreasing for $s \geq \beta$ as checked above, we obtain the assertion. □

Lemma 10.28. *For real numbers a, b and $\beta \geq 1$ and functions*

$$R \in \mathsf{DDE}(a, b, \beta),$$

assume that the following holds:

(i) *We have $a, b > 0$.*

(ii) *We have*

$$\begin{cases} R(s) \text{ is not constantly zero} \\ R(s) \geq 0 \end{cases} \quad \text{for } \beta - 1 < s < \beta$$

(iii) *The Iwaniec pairing vanishes, i.e.* $\langle R, r\rangle(s) = 0$ *with* $r(s) := r_{a,b}(s)$.
(iv) *We have* $r(s) > 0$ *for* $s \geq \beta$.

We have

$$\begin{cases} W_+(s) := R(s)e^{\phi_+(s)} \text{ is increasing} \\ W_-(s) := R(s)e^{\phi_-(s)} \text{ is decreasing} \end{cases}$$

for $s \geq \beta$ *with some sufficiently large* $c_\pm = c_\pm(R) \geq 1$, *where*

$$\phi_\pm(s) := \int_b^s \xi\left(\frac{t}{b}\right)dt \pm c_\pm s.$$

Proof. Write

$$\lambda := a + b > 0.$$

Note that Lemma 10.15 and the assumptions (i), (ii), (iii) and (iv) imply that $R(s) > 0$ for $s \geq \beta$. We first give some preliminary estimate. We shall use $s_0 = s_0(R)$ to assure various s-variables are large in the sense $s \geq s_0$. However, in this proof, the implicit constants and the constant s_0 may depend on R but should be independent of $c_\pm$. By Proposition 10.6, we have

$$(sr(s))' = ar(s) + br(s+1) = s^{\lambda-1}\left(\lambda + \left(\frac{1}{s}\right)\right)$$

$$r'(s) = \frac{sr'(s)}{s} = \frac{(sr(s))' - r(s)}{s} = s^{\lambda-2}\left(\lambda - 1 + \left(\frac{1}{s}\right)\right).$$

Therefore, we have

$$(10.41)\qquad \begin{aligned}(\log r(s+1))' &= \frac{r'(s+1)}{r(s+1)} = \frac{(s+1)r'(s+1)}{(s+1)r(s+1)} \\ &= \frac{(a-1)r(s+1) + br(s+2)}{(s+1)r(s+1)} = \frac{\lambda - 1}{s} + O\left(\frac{1}{s^2}\right)\end{aligned}$$

and

$$\begin{aligned}(\log r(s+1))'' &= \left(\frac{(a-1)r(s+1) + br(s+2)}{sr(s+1)}\right)' \\ &= \left(\frac{a-1}{s} + b\frac{r(s+2)}{sr(s+1)}\right)' \\ &= -\frac{a-1}{s^2} + b\frac{sr'(s+2)r(s+1) - r(s+2)(sr(s+1))'}{(sr(s+1))^2} \\ &= -\frac{a-1}{s^2} + b\frac{(\lambda-1)s^{2\lambda-2} - \lambda s^{2\lambda-2}}{s^{2\lambda}} + O\left(\frac{1}{s}\cdot\frac{s^{2\lambda-2}}{s^{2\lambda}}\right) \\ &= -\frac{\lambda-1}{s^2} + O\left(\frac{1}{s^3}\right).\end{aligned}$$

so

$$(\log r(s+1))'' = -\frac{\lambda - 1}{s^2} + O\left(\frac{1}{s^3}\right). \tag{10.42}$$

For the derivatives of $\phi_\pm(s)$, we just have

$$\phi_\pm'(s) = \xi\left(\frac{s}{b}\right) \pm c_\pm \quad \text{and} \quad \phi_\pm''(s) = \frac{1}{b}\xi'\left(\frac{s}{b}\right). \tag{10.43}$$

Since

$$(\log R(s))' = \frac{R'(s)}{R(s)} = \frac{sR'(s)}{sR(s)} = -\frac{aR(s) + bR(s-1)}{sR(s)} = -\frac{bR(s-1)}{sR(s)} - \frac{a}{s},$$

we have

$$(\log W_\pm(s))' = \frac{R'(s)}{R(s)} + \phi_\pm'(s) = -\frac{bR(s-1)}{sR(s)} + \xi\left(\frac{s}{\lambda}\right) \pm c_\pm - \frac{a}{s}. \tag{10.44}$$

We now assume that $\widetilde{s} = \widetilde{s}_\pm \geq s_0(R) \geq \beta + 1$ satisfies

$$(\log W_\pm(\widetilde{s}))' = 0$$

(the value of $s_0(R)$ will be made large depending only on R) or, equivalently,

$$\frac{bR(\widetilde{s} - 1)}{\widetilde{s}R(\widetilde{s})} = \xi\left(\frac{\widetilde{s}}{b}\right) \pm c_\pm - \frac{a}{\widetilde{s}}. \tag{10.45}$$

By Lemma 10.27, we have

$$\xi\left(\frac{\widetilde{s}}{b}\right) \pm c_\pm \geq 1 + O\left(\frac{1}{\widetilde{s}}\right). \tag{10.46}$$

Write

$$\psi_\pm(t) := \phi_\pm(t) - \log r(t+1).$$

Our next preliminary estimate is an approximation of the integral

$$\int_{\widetilde{s}-1}^{\widetilde{s}} r(t+1)e^{-\phi_\pm(t)}dt = \int_{\widetilde{s}-1}^{\widetilde{s}} e^{-\psi_\pm(t)}dt.$$

By using (10.41), (10.43) and Taylor's theorem, for $\widetilde{s} - 1 \leq t \leq \widetilde{s}$, we have

$$\begin{aligned}\psi_\pm'(t) &= \xi\left(\frac{t}{b}\right) \pm c_\pm - \frac{\lambda - 1}{t} + O\left(\frac{1}{\widetilde{s}^2}\right) \\ &= \xi\left(\frac{\widetilde{s}}{b}\right) \pm c_\pm + \frac{t - \widetilde{s}}{b}\xi'\left(\frac{\widetilde{s}}{b}\right) + \frac{(t - \widetilde{s})^2}{2b^2}\xi''\left(\frac{\sigma}{b}\right) - \frac{\lambda - 1}{\widetilde{s}} + O\left(\frac{1}{\widetilde{s}^2}\right)\end{aligned}$$

with some $\sigma \in (\widetilde{s} - 1, \widetilde{s})$ and and so by Proposition 10.20, we have

$$\psi_\pm'(t) = \xi\left(\frac{\widetilde{s}}{b}\right) \pm c_\pm + \frac{t - \widetilde{s} - \lambda + 1}{\widetilde{s}} + O\left(\frac{1}{\widetilde{s}\log\widetilde{s}}\right). \tag{10.47}$$

By using (10.46), we have

$$\begin{aligned}\xi\left(\frac{\widetilde{s}}{b}\right) \pm c_\pm + \frac{t-\widetilde{s}-\lambda+1}{\widetilde{s}} &= \xi\left(\frac{\widetilde{s}}{b}\right) \pm c_\pm + O\left(\frac{1}{\widetilde{s}}\right) \\ &= \left(\xi\left(\frac{\widetilde{s}}{b}\right) \pm c_\pm\right)\left(1+O\left(\frac{1}{\widetilde{s}(\xi(\frac{\widetilde{s}}{b}) \pm c_\pm)}\right)\right) \\ &= \left(\xi\left(\frac{\widetilde{s}}{b}\right) \pm c_\pm\right)\left(1+O\left(\frac{1}{\widetilde{s}}\right)\right) \\ &\asymp \xi\left(\frac{\widetilde{s}}{b}\right) \pm c_\pm\end{aligned} \tag{10.48}$$

for $\widetilde{s}-1 \le t \le \widetilde{s}$ since $\widetilde{s} \ge s_0(R)$ and so we obtain

$$\begin{aligned}&\psi'_\pm(t) \\ &= \left(\xi\left(\frac{\widetilde{s}}{b}\right) \pm c_\pm + \frac{t-\widetilde{s}-\lambda+1}{\widetilde{s}}\right)\left(1+O\left(\frac{1}{(\widetilde{s}\log\widetilde{s})(\xi(\frac{\widetilde{s}}{b}) \pm \frac{c_\pm}{\widetilde{s}})}\right)\right)\end{aligned} \tag{10.49}$$

for $\widetilde{s}-1 \le t \le \widetilde{s}$. By (10.46) and (10.48), we also have

$$\begin{aligned}\psi'_\pm(t) &= \left(\xi\left(\frac{\widetilde{s}}{b}\right) \pm c_\pm - \frac{\lambda}{\widetilde{s}}\right)\left(1+O\left(\frac{1}{\widetilde{s}(\xi(\frac{\widetilde{s}}{b}) \pm \frac{c_\pm}{\widetilde{s}})}\right)\right) \\ &\asymp \left(\xi\left(\frac{\widetilde{s}}{b}\right) \pm c_\pm - \frac{\lambda}{\widetilde{s}}\right) \asymp \left(\xi\left(\frac{\widetilde{s}}{b}\right) \pm c_\pm\right) > 0\end{aligned} \tag{10.50}$$

for $\widetilde{s}-1 \le t \le \widetilde{s}$ since $\widetilde{s} \ge s_0(R)$. By (10.42), (10.43) and Proposition 10.20,

$$\psi''_\pm(s) = \frac{1}{b}\xi'\left(\frac{s}{b}\right) - (\log r(s+1))'' \ll \frac{1}{s}. \tag{10.51}$$

By Taylor's theorem, we have

$$\psi_\pm(t) = \psi_\pm(\widetilde{s}-1) + (t-(\widetilde{s}-1))\psi'_\pm(\widetilde{s}-1) + \frac{(t-(\widetilde{s}-1))^2}{2}\psi''_\pm(\sigma)$$

for $\widetilde{s}-1 \le t \le \widetilde{s}$ with some $\sigma \in [\widetilde{s}-1, t]$. Thus, by (10.47) and (10.51),

$$\psi_\pm(t) = \psi_\pm(\widetilde{s}-1) + (t-(\widetilde{s}-1))\left(\xi\left(\frac{\widetilde{s}}{b}\right) \pm c_\pm\right) + O\left(\frac{1}{\widetilde{s}}\right) \tag{10.52}$$

for $\widetilde{s}-1 \le t \le \widetilde{s}$. By integration by parts with noting (10.50), we thus have

$$\begin{aligned}\int_{\widetilde{s}-1}^{\widetilde{s}} e^{-\psi_\pm(t)}dt &= -\int_{\widetilde{s}-1}^{\widetilde{s}} (e^{-\psi_\pm(t)})' \frac{dt}{\psi'_\pm(t)} \\ &= \frac{e^{-\psi_\pm(\widetilde{s}-1)}}{\psi'_\pm(\widetilde{s}-1)} - \frac{e^{-\psi_\pm(\widetilde{s})}}{\psi'_\pm(\widetilde{s})} - \int_{\widetilde{s}-1}^{\widetilde{s}} e^{-\psi_\pm(t)}\frac{\psi''_\pm(t)}{(\psi'_\pm(t))^2}dt.\end{aligned}$$

By (10.48) and (10.49), we have

$$\frac{e^{-\psi_\pm(\widetilde{s}-1)}}{\psi'_\pm(\widetilde{s}-1)} = \frac{e^{-\psi_\pm(\widetilde{s}-1)}}{\xi(\frac{\widetilde{s}}{b}) \pm c_\pm - \frac{\lambda}{\widetilde{s}}}\left(1+O\left(\frac{1}{(\widetilde{s}\log\widetilde{s})(\xi(\frac{\widetilde{s}}{b}) \pm \frac{c_\pm}{\widetilde{s}})}\right)\right).$$

By (10.46), (10.50) and (10.52), we have

$$\frac{e^{-\psi_\pm(\widetilde{s})}}{\psi'_\pm(\widetilde{s})} = \frac{e^{-\psi_\pm(\widetilde{s}-1)-\xi(\frac{\widetilde{s}}{b})\mp c_\pm+O(\frac{1}{\widetilde{s}})}}{\xi(\frac{\widetilde{s}}{b}) \pm c_\pm - \frac{\lambda}{\widetilde{s}}}.$$

Also, by (10.48), (10.50), (10.51) and (10.52), we have

$$\int_{\widetilde{s}-1}^{\widetilde{s}} e^{-\psi_\pm(t)} \frac{\psi''_\pm(t)}{(\psi'_\pm(t))^2} dt \ll \frac{e^{-\psi_\pm(\widetilde{s}-1)}}{\widetilde{s}(\xi(\frac{\widetilde{s}}{b}) \pm c_\pm)^2} \int_{\widetilde{s}-1}^{\widetilde{s}} e^{-(t-(\widetilde{s}-1))(\xi(\frac{\widetilde{s}}{b})\pm c_\pm)} dt$$
$$\ll \frac{e^{-\psi_\pm(\widetilde{s}-1)}}{\widetilde{s}(\xi(\frac{\widetilde{s}}{b}) \pm c_\pm - \frac{\lambda}{\widetilde{s}})(\xi(\frac{\widetilde{s}}{b}) \pm c_\pm)^2}.$$

By combining the above arguments, we have

$$\int_{\widetilde{s}-1}^{\widetilde{s}} e^{-\psi_\pm(t)} dt = \frac{e^{-\psi_\pm(\widetilde{s}-1)}}{\xi(\frac{\widetilde{s}}{b}) \pm c_\pm - \frac{\lambda}{\widetilde{s}}} \Big(1 - e^{-\xi(\frac{\widetilde{s}}{b}) \mp c_\pm + O(\frac{1}{\widetilde{s}})} + E(\widetilde{s})\Big).$$

with

$$E(\widetilde{s}) \ll \frac{1}{(\widetilde{s}\log \widetilde{s})(\xi(\frac{\widetilde{s}}{b}) \pm c_\pm)} + \frac{1}{\widetilde{s}(\xi(\frac{\widetilde{s}}{b}) \pm c_\pm)^2}.$$

By the definition of $\xi(\frac{\widetilde{s}}{\lambda})$ and (i) of Proposition 10.20, we have

$$e^{-\xi(\frac{\widetilde{s}}{b}) \mp c_\pm} = \frac{\xi(\frac{\widetilde{s}}{b})}{e^{\xi(\frac{\widetilde{s}}{b})} - 1} \frac{e^{\mp c_\pm + O(\frac{1}{\widetilde{s}})}}{\xi(\frac{\widetilde{s}}{b})} = \frac{b}{\widetilde{s}} \frac{e^{\mp c_\pm + O(\frac{1}{\widetilde{s}})}}{\xi(\frac{\widetilde{s}}{b})}.$$

Therefore, we obtain

$$\int_{\widetilde{s}-1}^{\widetilde{s}} e^{-\psi_\pm(t)} dt = \frac{e^{-\psi_\pm(\widetilde{s}-1)}}{\xi(\frac{\widetilde{s}}{b}) \pm c_\pm - \frac{\lambda}{\widetilde{s}}} \Bigg(1 - \frac{b}{\widetilde{s}} \frac{e^{\mp c_\pm + O(\frac{1}{\widetilde{s}})}}{\xi(\frac{\widetilde{s}}{b})} + E(\widetilde{s})\Bigg). \tag{10.53}$$

We now prove the assertion. We first consider $W_+(s)$. By (10.44), by taking c_+ sufficiently large, we can make $(\log W_+(s))' > 0$ for $\beta \le s \le s_0$ for sufficiently large $s_0 = s_0(R) \ge \beta + 1$. We assume to the contrary to the assertion that $W_+(s)$ is not increasing. By the continuity of $(\log W_+(s))'$, we can then take the least $\widetilde{s} \ge s_0$ with $(\log W_+(\widetilde{s}))' \le 0$ and such chosen $\widetilde{s}$ satisfies the condition (10.45). We then have

$$W_+(s) \text{ is increasing for } \beta \le s \le \widetilde{s}.$$

By (i), (iii) and (iv), we have

$$\begin{aligned}
\widetilde{s}r(\widetilde{s})R(\widetilde{s}) &= b\int_{\widetilde{s}-1}^{\widetilde{s}} r(t+1)R(t)dt \\
&= b\int_{\widetilde{s}-1}^{\widetilde{s}} r(t+1)W_+(t)e^{-\phi_+(t)}dt \qquad (10.54)\\
&\ge bW_+(\widetilde{s}-1)\int_{\widetilde{s}-1}^{\widetilde{s}} e^{-\psi_+(t)}dt.
\end{aligned}$$

By using (10.53) and (10.54),

$$\begin{aligned}
\widetilde{s}r(\widetilde{s})R(\widetilde{s}) &\ge \frac{bW_+(\widetilde{s}-1)e^{-\psi_+(\widetilde{s}-1)}}{\xi(\frac{\widetilde{s}}{b}) + c_+ - \frac{\lambda}{\widetilde{s}}} \Bigg(1 - \frac{b}{\widetilde{s}} \frac{e^{-c_+ + O(\frac{1}{\widetilde{s}})}}{\xi(\frac{\widetilde{s}}{b})} + E(\widetilde{s})\Bigg) \\
&= \frac{br(\widetilde{s})R(\widetilde{s}-1)}{\xi(\frac{\widetilde{s}}{b}) + c_+ - \frac{\lambda}{\widetilde{s}}} \Bigg(1 - \frac{b}{\widetilde{s}} \frac{e^{-c_+ + O(\frac{1}{\widetilde{s}})}}{\xi(\frac{\widetilde{s}}{b})} + E(\widetilde{s})\Bigg).
\end{aligned}$$

By using (10.45) and (10.48), we then have

$$
\begin{aligned}
1 &\geq \frac{\xi(\frac{\widetilde{s}}{b}) + c_+ - \frac{a}{\widetilde{s}}}{\xi(\frac{\widetilde{s}}{b}) + c_+ - \frac{\lambda}{\widetilde{s}}}\left(1 - \frac{b}{\widetilde{s}}\frac{e^{-c_+ + O(\frac{1}{\widetilde{s}})}}{\xi(\frac{\widetilde{s}}{b})} + E(\widetilde{s})\right)\\
&= \left(1 + \frac{b}{\widetilde{s}}\frac{1}{\xi(\frac{\widetilde{s}}{b}) + c_+ - \frac{\lambda}{\widetilde{s}}}\right)\left(1 - \frac{b}{\widetilde{s}}\frac{e^{-c_+ + O(\frac{1}{\widetilde{s}})}}{\xi(\frac{\widetilde{s}}{b})} + E(\widetilde{s})\right)\\
&\geq \left(1 + \frac{b}{\widetilde{s}}\frac{1}{\xi(\frac{\widetilde{s}}{b}) + c_+ - \frac{\lambda}{\widetilde{s}}}\right)\left(1 - \frac{b}{\widetilde{s}}\frac{e^{-\frac{c_+}{2}}}{\xi(\frac{\widetilde{s}}{b})} + E(\widetilde{s})\right)
\end{aligned}
\tag{10.55}
$$

since $\widetilde{s} \geq s_0(R)$. With the choice of the sign $\pm = +$, by Proposition 10.20, we have

$$
E(\widetilde{s}) \ll \frac{1}{(\widetilde{s}\log\widetilde{s})(\xi(\frac{\widetilde{s}}{b}) + c_+)} + \frac{1}{\widetilde{s}(\xi(\frac{\widetilde{s}}{b}) + c_+)^2} \ll \frac{1}{(\widetilde{s}\log\widetilde{s})(\xi(\frac{\widetilde{s}}{b}) + c_+)} \ll 1.
$$

Therefore, by (10.55), we have

$$
\begin{aligned}
1 &\geq 1 + \frac{b}{\widetilde{s}}\frac{1}{\xi(\frac{\widetilde{s}}{b}) + c_+ - \frac{\lambda}{\widetilde{s}}} - \frac{b}{\widetilde{s}}\frac{e^{-\frac{c_+}{2}}}{\xi(\frac{\widetilde{s}}{b})} + O\left(\frac{1}{(\widetilde{s}\log\widetilde{s})(\xi(\frac{\widetilde{s}}{b}) + c_+)}\right)\\
&\geq 1 + \frac{1}{2}\frac{b}{\widetilde{s}}\frac{1}{\xi(\frac{\widetilde{s}}{b}) + c_+ - \frac{\lambda}{\widetilde{s}}} - \frac{b}{\widetilde{s}}\frac{e^{-\frac{c_+}{2}}}{\xi(\frac{\widetilde{s}}{b})}
\end{aligned}
$$

since $\widetilde{s} \geq s_0(R)$. For $c_+ \geq 48$, since

$$
e^{-\frac{c_+}{2}} = \frac{1}{e^{\frac{c_+}{2}}} \leq \frac{8}{c_+^2} \leq \frac{16}{c_+}\frac{1}{1 + c_+} \leq \frac{16}{c_+}\frac{1}{1 + \frac{1}{\xi(\frac{\widetilde{s}}{b})}(c_+ - \frac{\lambda}{\widetilde{s}})} \leq \frac{1}{3}\frac{1}{1 + \frac{1}{\xi(\frac{\widetilde{s}}{b})}(c_+ - \frac{\lambda}{\widetilde{s}})}
$$

and $s_0 \geq s_0(R)$, we then have

$$
\begin{aligned}
1 &\geq 1 + \frac{1}{2}\frac{b}{\widetilde{s}}\frac{1}{\xi(\frac{\widetilde{s}}{b}) + c_+ - \frac{\lambda}{\widetilde{s}}} - \frac{1}{3}\frac{b}{\widetilde{s}}\frac{1}{\xi(\frac{\widetilde{s}}{b}) + c_+ - \frac{\lambda}{\widetilde{s}}}\\
&\geq 1 + \frac{1}{6}\frac{b}{\widetilde{s}}\frac{1}{\xi(\frac{\widetilde{s}}{b}) + c_+ - \frac{\lambda}{\widetilde{s}}},
\end{aligned}
$$

which is a contradiction. Thus, the assertion holds for $W_+(s)$.

We next consider $W_-(s)$. By (10.44), by taking c_- sufficiently large, we can make $(\log W_-(s))' < 0$ for $\beta \leq s \leq s_0$ for sufficiently large $s_0 = s_0(R)$. We assume to the contrary to the assertion that $W_-(s)$ is not decreasing. By the continuity of $(\log W_-(s))'$, we can then take the least $\widetilde{s} \geq s_0$ with $(\log W(\widetilde{s}))' \geq 0$ and such chosen $\widetilde{s}$ satisfies the condition (10.45). We then have

$$
W_-(s) \text{ is decreasing for } \beta \leq s \leq \widetilde{s}.
$$

By (i), (iii) and (iv), we have, we have

$$
\begin{aligned}
\widetilde{s}r(\widetilde{s})R(\widetilde{s}) &= b\int_{\widetilde{s}-1}^{\widetilde{s}} r(t+1)R(t)dt\\
&= b\int_{\widetilde{s}-1}^{\widetilde{s}} r(t+1)W_-(t)e^{-\phi_-(t)}dt\\
&\leq bW_-(\widetilde{s}-1)\int_{\widetilde{s}-1}^{\widetilde{s}} e^{\psi_-(t)}dt.
\end{aligned}
$$

By using (10.53), we then have

$$\begin{aligned}\widetilde{s}r(\widetilde{s})R(\widetilde{s}) &\le \frac{bW_-(\widetilde{s}-1)e^{-\psi_-(\widetilde{s}-1)}}{\xi(\frac{\widetilde{s}}{b})-c_--\frac{\lambda}{\widetilde{s}}}\left(1-\frac{b}{\widetilde{s}}\frac{e^{+c_-+O(\frac{1}{\widetilde{s}})}}{\xi(\frac{\widetilde{s}}{b})}+E(\widetilde{s})\right)\\ &= \frac{br(\widetilde{s})R(\widetilde{s}-1)}{\xi(\frac{\widetilde{s}}{b})-c_--\frac{\lambda}{\widetilde{s}}}\left(1-\frac{b}{\widetilde{s}}\frac{e^{+c_-+O(\frac{1}{\widetilde{s}})}}{\xi(\frac{\widetilde{s}}{b})}+E(\widetilde{s})\right).\end{aligned}$$

By using (10.45), we then have

$$\begin{aligned}1 &\le \frac{\xi(\frac{\widetilde{s}}{b})-c_--\frac{a}{\widetilde{s}}}{\xi(\frac{\widetilde{s}}{b})-c_--\frac{\lambda}{\widetilde{s}}}\left(1-\frac{b}{\widetilde{s}}\frac{e^{+c_-+O(\frac{1}{\widetilde{s}})}}{\xi(\frac{\widetilde{s}}{b})}+E(\widetilde{s})\right)\\ &= \left(1+\frac{b}{\widetilde{s}}\frac{1}{\xi(\frac{\widetilde{s}}{b})-c_--\frac{\lambda}{\widetilde{s}}}\right)\left(1-\frac{b}{\widetilde{s}}\frac{e^{+c_-+O(\frac{1}{\widetilde{s}})}}{\xi(\frac{\widetilde{s}}{b})}+E(\widetilde{s})\right) \qquad (10.56)\\ &\le \left(1+\frac{b}{\widetilde{s}}\frac{1}{\xi(\frac{\widetilde{s}}{b})-c_--\frac{\lambda}{\widetilde{s}}}\right)\left(1-\frac{b}{\widetilde{s}}\frac{e^{+\frac{c_-}{2}}}{\xi(\frac{\widetilde{s}}{b})}+E(\widetilde{s})\right).\end{aligned}$$

With the choice of the sign $\pm = -$, by Proposition 10.20 and (10.46), we have

$$E(\widetilde{s}) \ll \frac{1}{(\widetilde{s}\log\widetilde{s})(\xi(\frac{\widetilde{s}}{b})-c_-)}+\frac{1}{\widetilde{s}(\xi(\frac{\widetilde{s}}{b})-c_-)^2} \ll \frac{1}{\widetilde{s}(\xi(\frac{\widetilde{s}}{b})-c_-)^2} \ll 1.$$

Therefore, by (10.56), we have

$$1 \le 1+\frac{b}{\widetilde{s}}\frac{1}{\xi(\frac{\widetilde{s}}{b})-c_--\frac{\lambda}{\widetilde{s}}}-\frac{b}{\widetilde{s}}\frac{e^{+\frac{c_-}{2}}}{\xi(\frac{\widetilde{s}}{b})}+O\left(\frac{1}{\widetilde{s}(\xi(\frac{\widetilde{s}}{b})-c_-)^2}\right).$$

By (10.46), we have

$$\xi\left(\frac{\widetilde{s}}{b}\right) \ge c_-+\frac{1}{2} \quad \text{and} \quad \frac{\lambda}{\widetilde{s}} \le \frac{1}{3} \quad \text{for } \widetilde{s} \ge s_0(R)$$

and so by taking $c_- \ge 1152$ and using

$$\frac{1}{4}e^{\frac{c_-}{4}} \ge \frac{1}{4}\cdot\frac{1}{2}\left(\frac{c_-}{4}\right)^2 \ge 9c_- \ge 3+6c_-,$$

we have

$$\begin{aligned}\frac{1}{\xi(\frac{\widetilde{s}}{b})-c_-} &\le \frac{1}{\xi(\frac{\widetilde{s}}{b})-c_--\frac{\lambda}{\widetilde{s}}} \le \frac{1}{\xi(\frac{\widetilde{s}}{b})}\frac{1}{1-\frac{c_-+\frac{1}{3}}{\xi(\frac{\widetilde{s}}{\lambda})}}\\ &\le \frac{1}{\xi(\frac{\widetilde{s}}{b})}\frac{1}{1-\frac{c_-+\frac{1}{3}}{c_-+\frac{1}{2}}} = \frac{3+6c_-}{\xi(\frac{\widetilde{s}}{b})} \le \frac{1}{4}\frac{e^{\frac{c_-}{4}}}{\xi(\frac{\widetilde{s}}{b})} \quad \text{for } \widetilde{s} \ge s_0(R).\end{aligned}$$

Therefore, by Proposition 10.20, we have

$$\begin{aligned}1 &\le 1+\frac{1}{4}\frac{b}{\widetilde{s}}\frac{e^{\frac{c_-}{2}}}{\xi(\frac{\widetilde{s}}{b})}-\frac{1}{2}\frac{b}{\widetilde{s}}\frac{e^{\frac{c_-}{2}}}{\xi(\frac{\widetilde{s}}{b})}+O\left(\frac{e^{\frac{c_-}{2}}}{\widetilde{s}\xi(\frac{\widetilde{s}}{b})^2}\right)\\ &\le 1-\frac{1}{4}\frac{b}{\widetilde{s}}\frac{e^{\frac{c_-}{2}}}{\xi(\frac{\widetilde{s}}{b})}+O\left(\frac{e^{\frac{c_-}{2}}}{(\widetilde{s}\log\widetilde{s})\xi(\frac{\widetilde{s}}{b})}\right)\end{aligned}$$

$$\leq 1 - \frac{1}{8}\frac{\lambda}{\widetilde{s}}\frac{e^{\frac{c_-}{2}}}{\xi(\frac{\widetilde{s}}{b})}$$

since $\widetilde{s} \geq s_0(R)$. This is a contradiction. This completes the proof. □

Lemma 10.29. *For real numbers a, b and $\beta \geq 1$ and functions*

$$R \in \mathsf{DDE}(a, b, \beta),$$

assume that the following holds:

(i) *We have $a, b > 0$.*

(ii) *We have*

$$\begin{cases} R(s) \text{ is not constantly zero} \\ R(s) \geq 0 \end{cases} \quad \text{for } \beta - 1 < s < \beta$$

(iii) *The Iwaniec pairing vanishes, i.e. $\langle R, r\rangle(s) = 0$ with $r(s) := r_{a,b}(s)$.*

(iv) *We have $r(s) > 0$ for $s \geq \beta$.*

Then, we have

$$R(s-1) = \frac{1}{b}R(s)(s\log es)\exp\left(O\left(\frac{\log\log 3s}{\log 2s}\right)\right) \quad \text{for } s \geq \beta + 1,$$

where the implicit constant depends only on R.

Proof. Note that Lemma 10.15 and the assumptions (i), (ii), (iii) and (iv) imply that $R(s) > 0$ for $s \geq \beta$. Thus, by the continuity of $R(s)$, we may assume that s is sufficiently large, say $s \geq s_0(R)$.

We use Lemma 10.28. Under the notation of Lemma 10.28, we have

$$(\log W_+(s))' \geq 0 \quad \text{and} \quad (\log W_-(s))' \leq 0 \quad \text{for large } s.$$

By (10.44) in the proof of Lemma 10.28, this implies

$$\frac{bR(s-1)}{sR(s)} = \xi\left(\frac{s}{b}\right) + O(1).$$

By Proposition 10.20, this further implies that

$$\frac{bR(s-1)}{sR(s)} = (\log es)\left(1 + O\left(\frac{\log\log s}{\log s}\right)\right)$$

and so the assertion follows for large s. □

11. Convergence problem

In this section, we discuss the convergence of the series

$$T^{\pm}(s) := \sum_{\substack{n\geq 1 \\ n\equiv \nu_\pm \ (\mathrm{mod}\ 2)}} f_n(s)$$

defined in (9.5). We assume

$$\kappa > 0$$

throughout this section.

We first show that the convergence of the series $T^{\pm}(s)$ in the appropriate range $s \in I_\pm$ is independent of the variable s and solely depends on β.

Lemma 11.1. *For any $s_0 \in I_\pm$, the following are equivalent:*

(i) *The series $T^+(s_0)$ or the series $T^-(s_0)$ converge.*
(ii) *Both of the series $T^\pm(s)$ converge for any $s \in I_\pm$.*

Proof. The implication (ii) $\Longrightarrow$ (i) is obvious and so it suffices to prove the reversed implication (i) $\Longrightarrow$ (ii). Assume that the series $T^\pm(s_0)$ converges with $s_0 \in I_\pm$. By Proposition 9.2, for any positive integer N, we have

$$\begin{aligned} s_0^\kappa T^\pm(s_0) &\geq s_0^\kappa \sum_{\substack{2\leq n\leq N\\ n\equiv\nu_\pm \pmod 2}} f_n(s_0) \\ &\geq \int_{\max(s_0,\beta+\varepsilon_\pm)}^\infty \sum_{\substack{2\leq n\leq N\\ n\equiv\nu_\pm \pmod 2}} f_{n-1}(t-1)dt^\kappa \\ &= \int_{\max(s_0,\beta+\varepsilon_\pm)}^\infty \sum_{\substack{1\leq n\leq N-1\\ n\equiv\nu_\mp \pmod 2}} f_n(t-1)dt^\kappa \end{aligned}$$

By taking the limit $N \to \infty$ with the monotone convergence theorem, we have

$$\int_{\max(s_0,\beta+\varepsilon_\pm)}^\infty T^\mp(t-1)dt^\kappa \leq s_0^\kappa T^\pm(s_0) < +\infty.$$

We thus have the convergence of $T^\mp(s)$ for

$$s > \max(s_0, \beta+\varepsilon_\pm) - 1 = \max(s_0 - 1, \beta - \varepsilon_\mp)$$

By the induction, we find that $T^\mp(s)$ is convergent for

$$s > \max(s_0 - n, \beta - \varepsilon_\pm)$$

for any $n \in \mathbb{N}$ and so for $s > \beta - \varepsilon_\pm$. What remains is the convergence of $T^-(\beta)$. By Proposition 9.2, for any positive integer N, we have

$$\begin{aligned} &\beta^\kappa \sum_{\substack{2\leq n\leq N\\ n\equiv\nu_- \pmod 2}} f_n(\beta) - (\beta+1)^\kappa \sum_{\substack{2\leq n\leq N\\ n\equiv\nu_- \pmod 2}} f_n(\beta+1) \\ &= \int_\beta^{\beta+1} \sum_{\substack{2\leq n\leq N\\ n\equiv\nu_- \pmod 2}} f_{n-1}(t-1)dt^\kappa = \int_\beta^{\beta+1} \sum_{\substack{1\leq n\leq N-1\\ n\equiv\nu_+ \pmod 2}} f_n(t-1)dt^\kappa. \end{aligned}$$

However, since $f_n(s)$ is constant on $\beta - 1 < s \leq \beta + 1$ for odd n, we have

$$\begin{aligned} &\beta^\kappa \sum_{\substack{2\leq n\leq N\\ n\equiv\nu_- \pmod 2}} f_n(\beta) \\ &\leq (\beta+1)^\kappa \sum_{\substack{1\leq n\leq N-1\\ n\equiv\nu_+ \pmod 2}} f_n(\beta) + (\beta+1)^\kappa \sum_{\substack{2\leq n\leq N\\ n\equiv\nu_- \pmod 2}} f_n(\beta+1) \\ &\leq (\beta+1)^\kappa T^+(\beta) + (\beta+1)^\kappa T^-(\beta+1) \end{aligned}$$

and so

$$0 \leq \beta^\kappa T^-(\beta) \leq (\beta+1)^\kappa T^+(\beta) + (\beta+1)^\kappa T^-(\beta+1) < +\infty.$$

Thus, $T^-(s)$ converges even at $s = \beta$. This completes the proof. $\square$

In order to describe the range of β for which $T^{\pm}(s)$ converges, we introduce some auxiliary functions and parameters. We use two standard solutions

$$p(s) = p_\kappa(s) := r_{\kappa,-\kappa}(s) \quad \text{and} \quad q(s) = q_\kappa(s) := r_{\kappa,+\kappa}(s).$$

Recall that their defining equations are

$$\begin{cases} (sp(s))' = \kappa p(s) - \kappa p(s+1) \\ (sq(s))' = \kappa q(s) + \kappa q(s+1). \end{cases} \tag{11.1}$$

We may rewrite these equations as

$$\begin{cases} sp'(s) = (\kappa-1)p(s) - \kappa p(s+1) \\ sq'(s) = (\kappa-1)q(s) + \kappa q(s+1). \end{cases}$$

or

$$\begin{cases} (s^{1-\kappa}p(s))' = -\kappa s^{-\kappa}p(s+1) \\ (s^{1-\kappa}q(s))' = +\kappa s^{-\kappa}q(s+1). \end{cases} \tag{11.2}$$

The function $p(s)$ is positive for $s > 0$ while the function $q(s)$ may have zeros. Thus, let ρ be the largest zero

$$\rho = \rho_\kappa := \sup\{s \in (0,+\infty) \mid q_\kappa(s) = 0\},$$

where we use a convention $\rho_\kappa = 0$ if $q_\kappa(s)$ has no zero.

We prepare one observation:

Proposition 11.2. *For $s \in (\rho-1,\rho)\cap(0,+\infty)$, we have $q(s) < 0$. Consequently, if there is the second largest zero ρ_1 of $q(s)$, then we have $\rho_1 \le \rho - 1$.*

Proof. When $\kappa \le \frac{1}{2}$, by Proposition 10.14, $q(s)$ has no zero and so $\rho = 0$ and the assertion is vacuous. We thus can assume $\kappa > \frac{1}{2}$ and so by Proposition 10.14, ρ is a genuine zero of $q(s)$. By (11.1) with $s = \rho$, we have

$$\rho q'(\rho) = (\kappa-1)q(\rho) + \kappa q(\rho+1) = \kappa q(\rho+1) > 0$$

since $q(\rho+1) > 0$ by the maximality of ρ and Proposition 10.6. Thus, $q(s)$ is negative for s slightly smaller than ρ. Assume to the contrary to the assertion that $q(s)$ has a zero in $(\rho-1,\rho)\cap(0,+\infty)$. Since $q(s)$ is negative for s slightly smaller than ρ, we can take the second largest zero ρ_1 of $q(s)$ with $\rho_1 \in (\rho-1,\rho)$ and then $q(s) < 0$ for $s \in (\rho_1,\rho)$. By (11.1) with $s = \rho_1$, we then have

$$\rho_1 q'(\rho_1) = (\kappa-1)q(\rho_1) + \kappa q(\rho_1+1) = \kappa q(\rho_1+1) > 0$$

since $\rho_1 + 1 > \rho$ and so the maximality of ρ gives $q(\rho_1+1) > 0$. However, this implies $q(s)$ is positive for s slightly larger than ρ_1. This is a contradiction. □

We further consider the function

$$D(s) = D_\kappa(s) := (s-1)^{1-\kappa}p_\kappa(s-1)q_\kappa(s) + (s-1)^{1-\kappa}q_\kappa(s-1)p_\kappa(s)$$

defined on $(1,+\infty)$. When $\kappa \le \frac{1}{2}$, this function $D(s)$ can be defined even for $[1,+\infty)$ i.e. even at $s = 1$ by its limit value from right

$$D(1) := \lim_{\sigma\searrow 1}\Big((s-1)^{1-\kappa}p_\kappa(s-1)q_\kappa(s) + (s-1)^{1-\kappa}q_\kappa(s-1)p_\kappa(s)\Big). \tag{11.3}$$

Indeed, by Proposition 10.10, we have

$$(\sigma-1)^{1-\kappa}p_\kappa(\sigma-1) = (\sigma-1)^{1-\kappa}r_{\kappa,-\kappa}(\sigma-1) \to e^{-\kappa\gamma}\Gamma(1-\kappa) \qquad \text{as } \sigma \searrow 1$$

and so it suffices to consider $u(\sigma-1)$. When $\kappa=\frac{1}{2}$, by Proposition 10.8, we have

$$(\sigma-1)^{1-\kappa}q_\kappa(\sigma-1)=(\sigma-1)^{1-\kappa}r_{\kappa,\kappa}(\sigma-1)=(\sigma-1)^{1-\kappa}\to 0 \qquad \text{as } \sigma\searrow 1.$$

When $\kappa<\frac{1}{2}$ and so $2\kappa<1$, by Proposition 10.10, we have

$$(\sigma-1)^{1-\kappa}q_\kappa(\sigma-1)=(\sigma-1)^{1-\kappa}r_{\kappa,\kappa}(\sigma-1)\to \frac{e^{\kappa\gamma}\Gamma(1-\kappa)}{\Gamma(1-2\kappa)} \qquad \text{as } \sigma\searrow 1.$$

By combining the above observations, we conclude that (11.3) exists. Based on this observation, for simplicity, we use the abbreviations like

$$(\beta-1)^{1-\kappa}p_\kappa(\beta-1)=\lim_{s\searrow\beta}(\sigma-1)^{1-\kappa}p_\kappa(\sigma-1),$$
$$(\beta-1)^{1-\kappa}q_\kappa(\beta-1)=\lim_{s\searrow\beta}(\sigma-1)^{1-\kappa}q_\kappa(\sigma-1).$$

In this way, we consider $D(s)$ as a function on the interval

$$J=J(\kappa):=\begin{cases}(1,+\infty) & \text{if } \kappa>\frac{1}{2},\\ [1,+\infty) & \text{if } \kappa\le\frac{1}{2}.\end{cases}$$

The non-vanishing of $D(s)$ is as important as ρ in our convergence problem. We thus check that $D(s)$ has at most one zero in $(\rho,+\infty)$.

Lemma 11.3. *For $s>\rho$, we have*

$$\left(\frac{q(s)}{p(s)}\right)'>0.$$

Consequently, the function

$$\frac{q(s)}{p(s)}$$

is strictly increasing for $s\ge\rho$.

Proof. Follows by Lemma 10.16 since $p(s)=r_{\kappa,-\kappa}(s)$ and $q(s)=r_{\kappa,+\kappa}(s)$. □

Proposition 11.4. *The function*

$$\frac{D(s)}{p(s)}$$

is strictly increasing for $s\in J\cap(\rho,+\infty)$.

Proof. We have

$$\frac{D(s)}{p(s)}=(s-1)^{1-\kappa}p(s-1)\frac{q(s)}{p(s)}+(s-1)^{1-\kappa}q(s-1).$$

By (11.2), we have

$$\begin{aligned}&\frac{d}{ds}\left(\frac{D(s)}{p(s)}\right)\\ &=-\kappa(s-1)^{-\kappa}p(s)\frac{q(s)}{p(s)}+(s-1)^{1-\kappa}p(s-1)\left(\frac{q(s)}{p(s)}\right)'+\kappa(s-1)^{-\kappa}q(s)\\ &=(s-1)^{1-\kappa}p(s-1)\left(\frac{q(s)}{p(s)}\right)'>0\end{aligned}$$

for $s \in J \cap (\rho, +\infty)$ by Lemma 11.3. This completes the proof. □

We then define $\widetilde{\rho}$ to be the unique zero of $D(s)$ in $J \cap (\rho, +\infty)$ if such zero exists and $\widetilde{\rho} \coloneqq \max(\rho, 1)$ if otherwise. By Proposition 11.4 and

$$D(s) \sim 2s^{\kappa-1} \quad \text{as } s \to \infty$$

follows by (10.16), we find that

$$D(s) > 0 \iff s > \widetilde{\rho}$$

provided $s \in J \cap (\rho, +\infty)$.

Proposition 11.5. *We have $\rho \leq \widetilde{\rho} \leq \rho + 1$.*

Proof. The inequality $\widetilde{\rho} \leq \rho + 1$ immediately follows since if $s > \rho + 1$, then

$$p(s), q(s), p(s-1), q(s-1) > 0, \qquad \text{and so} \quad D(s) > 0.$$

Also, $\rho \leq \widetilde{\rho}$ follows by definition. This completes the proof. □

The following two results are mentioned without detailed proof in Kai-Man Tsang's paper.

Proposition 11.6. *If the series $T^{\pm}(s)$ converges for $s \in I_{\pm}$, then $\beta > \widetilde{\rho}$.*

Proof. Assume that the series $T^{\pm}(s)$ converges for $s \in I_{\pm}$. We then should have the properties of $T^{\pm}(s)$ proven in Proposition 9.4 and Proposition 9.4. We further use the functions $P(s), Q(s)$ defined by

$$\begin{cases} P(s) \coloneqq T^{+}(s) - T^{-}(s) + 2 \\ Q(s) \coloneqq T^{+}(s) + T^{-}(s) \end{cases} \quad \text{for } s \geq \beta$$

and

$$s^{\kappa} P(s) = s^{\kappa} Q(s) = A \quad \text{for } \beta - 1 < s < \beta$$

as in (10.2) and (10.5). We then have

$$\begin{cases} s^{\kappa} P(s) = A + B + A \displaystyle\int_{\beta}^{s} \frac{dt^{\kappa}}{(t-1)^{\kappa}} \\ s^{\kappa} Q(s) = A - B - A \displaystyle\int_{\beta}^{s} \frac{dt^{\kappa}}{(t-1)^{\kappa}} \end{cases} \quad \text{for } \beta \leq s \leq \beta + 1$$

and

$$\begin{cases} sP'(s) = -\kappa P(s) + \kappa P(s-1) \\ sQ'(s) = -\kappa Q(s) - \kappa Q(s-1) \end{cases} \quad \text{for } s \in (\beta, \beta+1) \cup (\beta+1, +\infty)$$

as seen in (10.4) and (10.6), where A, B are given by the equations

$$A = (\beta+1)^{\kappa} T^{+}(\beta+1) + (\beta+1)^{\kappa},$$

$$\beta^{\kappa} T^{-}(\beta) = \beta^{\kappa} - B$$

as in Proposition 9.4. Therefore, $p(s), q(s)$ are the adjoint solutions of $P(s), Q(s)$, respectively. By Lemma 10.1, we may write

$$C_P \coloneqq \langle P, p\rangle(s) = sp(s)P(s) + \kappa \int_{s-1}^{s} p(t+1)P(t)dt,$$
$$C_Q \coloneqq \langle Q, q\rangle(s) = sq(s)Q(s) - \kappa \int_{s-1}^{s} q(t+1)Q(t)dt$$

for $s > \beta$. We then prove

$$C_P = 2 \quad \text{and} \quad C_Q \geq 0.$$

By (v) of Proposition 9.4, we have $P(s) \to 2$ as $s \to \infty$. By Lemma 10.26, we have $P(s) \sim C_P$ as $s \to \infty$ and so $C_P = 2$. By (ii) of Proposition 9.4, $Q(s)$ is positive for $s \geq \beta$. Also, by definition, A should be positive and so by the definition of the extension of $Q(s)$, we find that $Q(s)$ is positive for $s > \beta - 1$. Thus, by Lemma 10.26, we should have $C_Q \geq 0$ since otherwise $Q(s)$ is negative for large s.

We first prove that $\beta > \rho$. Assume to the contrary that $\rho \geq \beta$. In this case, we have $\rho \neq 0$ and so ρ is a genuine zero of $q(s)$. For $s > \rho$,

$$0 \leq C_Q = sq(s)Q(s) - \kappa \int_{s-1}^{s} q(t+1)Q(t)dt$$

and so

$$\kappa \int_{\rho-\frac{1}{2}}^{\rho} q(t+1)Q(t)dt \leq \kappa \int_{s-1}^{s} q(t+1)Q(t)dt \leq sq(s)Q(s)$$

for $\rho < s \leq \rho + \frac{1}{2}$. Since $q(\rho) = 0$, by taking the limit $s \searrow \rho$, we find that

$$\kappa \int_{\rho-\frac{1}{2}}^{\rho} q(t+1)Q(t)dt \leq 0$$

However, by the maximality of ρ, we have $q(t+1) > 0$ in the above integral and also $Q(t) > 0$ in the same integral as seen above. This implies

$$0 < \kappa \int_{s-1}^{s} q(t+1)Q(t)dt \leq sq(s)Q(s),$$

which is a contradiction. Therefore, we should have $\beta > \rho$.

We now prove $\beta > \widetilde{\rho}$. By the previous paragraph, we may assume $\beta > \rho$ and so, by the maximality of ρ, we have $q(s) > 0$ for $s \geq \beta$. By Lemma 10.2 and

$$P(s) = \frac{A}{s^{\kappa}} \quad \text{for } \beta - 1 < s < \beta,$$

we have

$$2 = C_P = \beta p(\beta)(P(\beta) - A\beta^{-\kappa}) + A(\beta-1)^{1-\kappa} p(\beta-1).$$

By the formula

$$s^{\kappa} P(s) = A + B + A \int_{\beta}^{s} \frac{dt^{\kappa}}{(t-1)^{\kappa}} \quad \text{for } \beta \leq s \leq \beta + 1,$$

we may rewrite the above formula as

$$2 = C_P = \beta^{1-\kappa} p(\beta)B + A(\beta-1)^{1-\kappa} p(\beta-1).$$

Therefore, we have

$$B = \frac{\beta^{\kappa-1}}{p(\beta)}\Big(2 - A(\beta-1)^{1-\kappa}p(\beta-1)\Big). \tag{11.4}$$

(Note that $p(\beta) \neq 0$.) By Lemma 10.2, we have

$$0 \le C_Q = \beta q(\beta)(Q(\beta) - A\beta^{-\kappa}) + A(\beta-1)^{1-\kappa}q(\beta-1).$$

Using the formula

$$s^\kappa Q(s) = A - B - A\int_\beta^s \frac{dt^\kappa}{(t-1)^\kappa} \quad \text{for } \beta \le s \le \beta+1,$$

we then have

$$0 \le C_Q = -\beta^{1-\kappa}q(\beta)B + A(\beta-1)^{1-\kappa}q(\beta-1).$$

By substituting (11.4), we obtain

$$0 \le -\frac{q(\beta)}{p(\beta)}\big(2 - A(\beta-1)^{1-\kappa}p(\beta-1)\big) + A(\beta-1)^{1-\kappa}q(\beta-1)$$

and so

$$\begin{aligned} 2\frac{q(\beta)}{p(\beta)} &\le A\bigg(\frac{q(\beta)}{p(\beta)}(\beta-1)^{1-\kappa}p(\beta-1) + (\beta-1)^{1-\kappa}q(\beta-1)\bigg) \\ &= A\frac{(\beta-1)^{1-\kappa}p(\beta-1)q(\beta) + (\beta-1)^{1-\kappa}q(\beta-1)p(\beta)}{p(\beta)} = A\frac{D(\beta)}{p(\beta)}. \end{aligned}$$

Since $\beta > \rho$, this gives

$$D(\beta) \ge \frac{2}{A}q(\beta) > 0$$

which implies $\beta > \widetilde{\rho}$ since $\beta \in J$. This completes the proof. □

Lemma 11.7. *For $t \ge \rho$, the function*

$$s \mapsto s^{1-\kappa}p(s)q(t) + s^{1-\kappa}q(s)p(t)$$

is strictly increasing for $s \in (0,+\infty) \cap [t-1,+\infty)$.

Proof. By (11.2), we have

$$(s^{1-\kappa}p(s)q(t) + s^{1-\kappa}q(s)p(t))' = \kappa s^{-\kappa}\Big(q(s+1)p(t) - p(s+1)q(s)\Big).$$

By Lemma 11.3, we have

$$q(s+1)p(t) - p(s+1)q(s) = p(t)p(s+1)\left(\frac{q(s+1)}{p(s+1)} - \frac{q(t)}{p(t)}\right) > 0$$

for $s+1 > t \ge \rho$. Thus, we obtain the lemma. □

Proposition 11.8. *Assume that $\kappa > 0$ and $\beta \in J$ satisfies $\beta > \widetilde{\rho}$. Then, the series $T^\pm(s)$ are convergent for $s \in I_\pm$ and also we have*

(i) *We have*

$$|P(s) - 2|, Q(s), T^\pm(s) \ll \exp(-s\log s - s\log\log s + s\log e\kappa)$$

for $s \ge \max(e^e, \beta)$, where the implicit constant depends only on κ, β.

(ii) *We have*

$$\begin{cases} A = \dfrac{2q(\beta)}{D(\beta)} > 0, \\ \\ B = \dfrac{2(\beta-1)^{1-\kappa} q(\beta-1)}{\beta^{1-\kappa} D(\beta)}. \end{cases}$$

for the constants A, B *given in Proposition 9.4. Also, we have*

$$A > (\beta+1)^{\kappa}.$$

(iii) *For the Iwaniec pairing, we have*

$$\langle P, p\rangle(s) = 2 \quad \text{and} \quad \langle Q, q\rangle(s) = 0 \quad \text{for } s \geq \beta.$$

(iv) *We have*

$$T^{\pm}(s) \asymp Q(s) > 0 \quad \text{for } s \geq \beta$$

and

$$0 \leq T^{\pm}(s) \leq Q(s) \quad \text{for } s \in I_{\pm},$$

where the implicit constant depends only on κ *and* β.

Proof. We prepare the auxiliary constants and functions

$$\widetilde{A}, \quad \widetilde{B}, \quad \widetilde{T}^{\pm}(s), \quad \widetilde{P}(s), \quad \widetilde{Q}(s),$$

which are turned out to coincide with the original constants and functions

$$A, \quad B, \quad T^{\pm}(s), \quad P(s), \quad Q(s).$$

We let

$$\begin{cases} \widetilde{A} \coloneqq \dfrac{2q(\beta)}{D(\beta)}, \\ \\ \widetilde{B} \coloneqq \left(\dfrac{\beta-1}{\beta}\right)^{1-\kappa} \dfrac{2q(\beta-1)}{D(\beta)}, \end{cases}$$

which are well-defined since $p(\beta), D(\beta) > 0$ by the assumption $\beta > \widetilde{\rho}$. By tracing the definition of $T^{\pm}(s)$, we define the continuous functions $\widetilde{T}^{\pm}(s)$ for $s \in I_{\pm}$ by the initial conditions

$$\begin{cases} s^{\kappa}\widetilde{T}^{+}(s) \coloneqq \widetilde{A} - s^{\kappa} \quad \text{for } \beta - 1 < s \leq \beta + 1, \\ \\ \beta^{\kappa}\widetilde{T}^{-}(\beta) \coloneqq \beta^{\kappa} - \widetilde{B} \end{cases} \tag{11.5}$$

and the delay–differential equation

$$(s^{\kappa}\widetilde{T}^{\pm}(s))' = -\kappa s^{\kappa-1}\widetilde{T}^{\mp}(s-1) \quad \text{for } s > \beta + \varepsilon_{\pm}. \tag{11.6}$$

We then define $\widetilde{U}(s), \widetilde{V}(s)$ by

$$\begin{cases} \widetilde{P}(s) \coloneqq \widetilde{T}^{+}(s) - \widetilde{T}^{-}(s) + 2 \\ \widetilde{Q}(s) \coloneqq \widetilde{T}^{+}(s) + \widetilde{T}^{-}(s) \end{cases} \quad \text{for } s \geq \beta$$

and

$$s^{\kappa}\widetilde{P}(s) = s^{\kappa}\widetilde{Q}(s) = \widetilde{A} \quad \text{for } \beta - 1 < s < \beta.$$

By arguing similarly to the proof of Proposition 11.6, we have

$$\begin{cases} \langle \widetilde{P}, p\rangle(s) = +\beta^{1-\kappa}p(\beta)\widetilde{B} + \widetilde{A}(\beta-1)^{1-\kappa}p(\beta-1) \\ \langle \widetilde{Q}, q\rangle(s) = -\beta^{1-\kappa}q(\beta)\widetilde{B} + \widetilde{A}(\beta-1)^{1-\kappa}q(\beta-1). \end{cases}$$

By substituting our choice of $\widetilde{A}, \widetilde{B}$, we get

$$\langle \widetilde{P}, p\rangle(s) = \frac{2(\beta-1)^{1-\kappa}q(\beta-1)p(\beta)}{D(\beta)} + \frac{2(\beta-1)^{1-\kappa}p(\beta-1)q(\beta)}{D(\beta)} = 2$$

and

$$\langle \widetilde{Q}, q\rangle(s) = -\frac{2(\beta-1)^{1-\kappa}q(\beta-1)q(\beta)}{D(\beta)} + \frac{2(\beta-1)^{1-\kappa}q(\beta-1)q(\beta)}{D(\beta)} = 0$$

so that

$$\langle \widetilde{P}, p\rangle(s) = 2 \quad \text{and} \quad \langle \widetilde{Q}, q\rangle(s) = 0 \quad \text{for } s \geq \beta. \tag{11.7}$$

Since the adjoint equation (11.1) gives

$$sp(s) = -\kappa \int_{s-1}^{s} p(t+1)dt + (\text{constant}) \quad \text{for } s \geq \beta$$

and then Proposition 10.6 shows

$$sp(s) = -\kappa \int_{s-1}^{s} p(t+1)dt + 1 \quad \text{for } s \geq \beta$$

Thus, we can rewrite (11.7) as

$$\langle \widetilde{P} - 2, p\rangle(s) = \langle \widetilde{Q}, q\rangle(s) = 0 \quad \text{for } s \geq \beta. \tag{11.8}$$

We next prove the following claim.

Claim 11.9.

(a) *We have*

$$|\widetilde{P}(s) - 2| < \widetilde{Q}(s) \qquad \text{for } s > \beta - 1.$$

Moreover, there exists a real number $\eta = \eta(\kappa, \beta) \in (0,1)$ *such that*

$$|\widetilde{P}(s) - 2| \leq \eta\widetilde{Q}(s) \quad \text{for } s \geq \beta.$$

Consequently, $\widetilde{T}^{\pm}(s) > 0$ *for* $s \in I_{\pm}$.

(b) *We have*

$$|\widetilde{P}(s) - 2|, \widetilde{Q}(s), \widetilde{T}^{\pm}(s) \ll \exp(-s\log s - s\log\log s + s\log e\kappa)$$

for $s > \max(e^e, \beta)$.

(c) *We have*

$$s^{\kappa}\widetilde{T}^{\pm}(s) = \int_s^{\infty} \widetilde{T}^{\mp}(t-1)dt^{\kappa} \quad \text{for } s \geq \beta + \varepsilon_{\pm}.$$

Proof.

(a) The positivity of $\widetilde{T}^{\pm}(s)$ follows from $|\widetilde{P}(s) - 2| < \widetilde{Q}(s)$ since then

$$\widetilde{T}^{\pm}(s) = \frac{1}{2}\big(\pm(\widetilde{P}(s) - 2) + \widetilde{Q}(s)\big) \geq \frac{1}{2}(\widetilde{Q}(s) - |\widetilde{P}(s) - 2|) > 0.$$

Thus, we prove $|\widetilde{P}(s)-2|<\widetilde{Q}(s)$ for $s>\beta-1$ and there exists $\eta\in(0,1)$ such that $|\widetilde{P}(s)-2|\le\eta\widetilde{Q}(s)$ for $s\ge\beta$. We want to use Lemma 10.17 with

$$a=b=\kappa,\quad R^{-}(s)=\widetilde{P}(s)-2,\quad R^{+}(s)=\widetilde{Q}(s).$$

To this end, we need to check $|\widetilde{P}(s)-2|<\widetilde{Q}(s)$ for $\beta-1<s<\beta$. Clearly,

$$\widetilde{P}(s)-2=\frac{\widetilde{A}}{s^{\kappa}}-2<\frac{\widetilde{A}}{s^{\kappa}}=\widetilde{Q}(s)\quad\text{for } \beta-1<s<\beta.$$

We thus prove the inequality

$$-(\widetilde{P}(s)-2)<\widetilde{Q}(s)\iff s^{\kappa}<\widetilde{A}$$

for $\beta-1<s<\beta$. It suffices to show

$$\widetilde{A}=\frac{2q(\beta)}{D(\beta)}>\beta^{\kappa}. \tag{11.9}$$

For $s>\beta$, by using Lemma 11.7 with $t=s$, we have

$$\begin{aligned}D(s)&=(s-1)^{1-\kappa}p(s-1)q(s)+(s-1)^{1-\kappa}q(s-1)p(s)\\&\le 2s^{1-\kappa}p(s)q(s).\end{aligned}$$

By taking the limit $s\searrow\beta$, we obtain

$$D(\beta)\le 2\beta^{1-\kappa}p(\beta)q(\beta).$$

Therefore, we have

$$\widetilde{A}=\frac{2q(\beta)}{D(\beta)}\ge\frac{\beta^{\kappa}}{\beta p(\beta)}.$$

By using the bound

$$p(s)=\int_0^\infty e^{-sx-\kappa\,\mathrm{Ein}(x)}dx<\int_0^\infty e^{-sx}dx=\frac{1}{s},$$

we obtain (11.9). This proves $|\widetilde{P}(s)-2|<\widetilde{Q}(s)$ for $\beta-1<s<\beta$. Then, (a) follows by Lemma 10.17 with recalling (11.8).

(b) By Lemma 10.24 and (11.8), the assertion immediately follows for $|\widetilde{P}(s)-2|$ and $\widetilde{Q}(s)$. For $\widetilde{T}^{\pm}(s)$, we can use

$$\widetilde{T}^{\pm}(s)=\frac{1}{2}\big(\pm(\widetilde{P}(s)-2)+\widetilde{Q}(s)\big).$$

This proves (b) of the claim.

(c) By the delay-differential equation (11.6) defining $\widetilde{T}^{\pm}(s)$, we have

$$s^{\kappa}\widetilde{T}^{\pm}(s)-\sigma^{\kappa}\widetilde{T}^{\pm}(\sigma)=\int_s^\sigma T^{\mp}(t-1)dt^{\kappa}\quad\text{for } \beta+\varepsilon_{\pm}\le s\le\sigma.$$

By taking the limit $\sigma\to\infty$ with (b), we obtain (c) of the claim. □

We now prove the convergence of $T^{\pm}(s)$. We consider the partial sum

$$T_N(s):=\sum_{\substack{1\le n\le N\\ n\equiv N\ (\mathrm{mod}\ 2)}}f_n(s)$$

as defined in (9.6). Note that by (ii) and (iii) of Proposition 9.3, we have

$$s^\kappa T_N(s) = \int_s^\infty T_{N-1}(t-1)dt^\kappa \quad \text{for } s > \beta + \varepsilon_N. \tag{11.10}$$

Since the terms of $T_N(s)$ are positive and (b) of Claim holds, it suffices to prove

$$\begin{cases} 0 \le s^\kappa T_N(s) \le s^\kappa \widetilde{T}^+(s) & \text{for } s \in I_+ \text{ and odd } N \ge 1, \\ 0 \le s^\kappa T_N(s) \le s^\kappa \widetilde{T}^-(s) & \text{for } s \in I_- \text{ and even } N \ge 1. \end{cases} \tag{11.11}$$

We use induction on N.

Initial case $N = 1$. By the initial condition (11.5) and (a) of Claim, we have

$$0 < (\beta+1)^\kappa \widetilde{T}^+(\beta+1) = \widetilde{A} - (\beta+1)^\kappa \quad \text{and so} \quad \widetilde{A} > (\beta+1)^\kappa.$$

Thus, we have

$$0 \le s^\kappa T_1(s) = (\beta+1)^\kappa - s^\kappa < \widetilde{A} - s^\kappa = s^\kappa \widetilde{T}^+(s) \quad \text{for } \beta - 1 < s \le \beta + 1.$$

Since $T_1(s) = 0$ for $s \ge \beta+1$, by the positivity of $\widetilde{T}^+(s)$, the same inequality holds even for $s \ge \beta + 1$. This proves (11.11) for the initial case $N = 1$.

Induction step from $N-1$ to N with even $N \ge 1$. Assume that $N \ge 1$ is even and (11.11) holds for $N-1$. By (11.10) and the induction hypothesis, we have

$$s^\kappa T_N(s) = \int_s^\infty T_{N-1}(t-1)dt^\kappa \le \int_s^\infty \widetilde{T}^+(t-1)dt^\kappa \quad \text{for } s > \beta$$

since $t - 1 \ge s - 1 > \beta - 1$ in the integral. By (c) of Claim, we have

$$0 \le s^\kappa T_N(s) \le s^\kappa \widetilde{T}^-(s) \quad \text{for } s > \beta.$$

By taking the limit, we can see this inequality holds even at $s = \beta$. Thus, (11.11) holds for the N-th case as well.

Induction step from $N-1$ to N with odd $N \ge 1$. Assume that $N \ge 1$ is odd and (11.11) holds for $N-1$. We first consider the range $s \ge \beta + 1$. By (11.10) and the induction hypothesis, we have

$$s^\kappa T_N(s) = \int_s^\infty T_{N-1}(t-1)dt^\kappa \le \int_s^\infty \widetilde{T}^-(t-1)dt^\kappa \quad \text{for } s \ge \beta + 1$$

since $t - 1 \ge s - 1 \ge \beta$ in the integral. By (c) of Claim, we have

$$0 \le s^\kappa T_N(s) \le s^\kappa \widetilde{T}^+(s) \quad \text{for } s \ge \beta + 1.$$

We next consider the range $\beta - 1 < s \le \beta + 1$. By (iv) of Proposition 9.3, we have

$$s^\kappa T_N(s) = (\beta+1)^\kappa T_N(\beta+1) + (\beta+1)^\kappa - s^\kappa \quad \text{for } \beta - 1 < s \le \beta + 1.$$

Since we have already shown (11.11) for the current N and $s = \beta + 1$, we have

$$s^\kappa T_N(s) \le (\beta+1)^\kappa \widetilde{T}^+(\beta+1) + (\beta+1)^\kappa - s^\kappa \quad \text{for } \beta - 1 < s \le \beta + 1.$$

By the initial condition (11.5) of $\widetilde{T}^+(s)$, we have

$$\begin{aligned} s^\kappa T_N(s) &\le \widetilde{A} - (\beta+1)^\kappa + (\beta+1)^\kappa - s^\kappa \\ &= \widetilde{A} - s^\kappa = s^\kappa \widetilde{T}^+(s) \quad \text{for } \beta - 1 < s \le \beta + 1. \end{aligned}$$

This completes the proof of (11.11) for the N-th case and so proves the convergence of $T^\pm(s)$.

Finally, we prove (i), (ii), (iii) and (iv). By (11.11) and (b) of Claim, we have

$$0 \le T^{\pm}(s) \le \widetilde{T}^{\pm}(s) \le \widetilde{Q}(s) \quad \text{for } s \ge \beta$$

and so

$$T^{\pm}(s) \ll \exp(-s\log s - s\log\log s + s\log e\kappa) \quad \text{for } s > \max(e^e, \beta).$$

Therefore, we should have

$$|P(s) - 2|, Q(s) \ll \exp(-s\log s - s\log\log s + s\log e\kappa)$$

for $s > \max(e^e, \beta)$. Thus, (i) follows. By Lemma 10.26, we should have (iii), i.e.

$$\langle P, p\rangle(s) = 2 \quad \text{and} \quad \langle Q, q\rangle(s) = 0 \quad \text{for } s > \beta.$$

By arguing similarly to the proof of Proposition 11.6, we can see

$$2 = \langle P, p\rangle(s) = +\beta^{1-\kappa}p(\beta)B + A\lim_{\sigma\searrow\beta} p(\sigma-1)(\sigma-1)^{1-\kappa},$$
$$0 = \langle Q, q\rangle(s) = -\beta^{1-\kappa}q(\beta)B + A\lim_{\sigma\searrow\beta} q(\sigma-1)(\sigma-1)^{1-\kappa}.$$

By solving this system of equations, we obtain the first assertions of (ii) and so

$$A = \widetilde{A}, \quad B = \widetilde{B}, \quad P(s) = \widetilde{P}(s), \quad Q(s) = \widetilde{Q}(s).$$

The inequality $A > (\beta+1)^{\kappa}$ can be derived by the positivity of $T^{+}(s)$ as

$$0 < (\beta+1)^{\kappa}T^{+}(\beta+1) = A - (\beta+1)^{\kappa}$$

This completes the proof of (ii). Then, for $s \ge \beta$, (a) of Claim implies

$$\begin{aligned} T^{\pm}(s) \le Q(s) &= \frac{1}{1-\eta}(Q(s) - \eta Q(s)) \\ &\le \frac{1}{1-\eta}(Q(s) \pm |P(s) - 2|) = \frac{2}{1-\eta}T^{\pm}(s) \end{aligned}$$

and

$$Q(s) > 0 \quad \text{for } s \ge \beta$$

and so (iv) follows provided $s \ge \beta$. When $\beta - 1 < s < \beta$ and $\pm = +$, we have

$$T^{+}(s) = \frac{A}{s^{\kappa}} - 1 \le \frac{A}{s^{\kappa}} = Q(s)$$

and so (iv) holds even for $\beta - 1 < s < \beta$. This completes the proof. ∎

Lemma 11.10. *For $s \ge \beta > \widetilde{\rho}$ with $\beta \in J$, we have the following*:

(i) *For $s \ge \beta$, we have*

$$T^{\pm}(s) \ll \exp(-s\log s - s\log\log 3s + s\log e\kappa)$$

and

$$T^{\pm}(s) = \exp\left(-s\log s - s\log\log 3s + s\log e\kappa + O\left(\frac{s\log\log 3s}{\log 2s}\right)\right),$$

where the implicit constant depends only on κ and β.

(ii) *For $s \ge \beta + 1$, we have*

$$T^{\pm}(s-1),\ T^{\mp}(s-1) \asymp T^{\pm}(s)(s\log es),$$

where the implicit constant depends only on κ and β.

Proof. By (iv) of Proposition 11.8, it suffices to prove the assertion for $Q(s)$, i.e.

$$Q(s) \ll \exp(-s\log s - s\log\log 3s + s\log e\kappa),$$

$$Q(s) = \exp\left(-s\log s - s\log\log 3s + s\log e\kappa + O\left(\frac{s\log\log 3s}{\log 2s}\right)\right)$$

and

$$Q(s-1) \asymp Q(s)(s\log es).$$

Recall that $\langle Q, q\rangle(s) = 0$ as in the proof of Proposition 11.8. Also, by (10.5), (ii) of Proposition 11.8 and $\beta > \widetilde{\rho}$, we have $Q(s) > 0$ for $\beta - 1 < s < \beta$. By Lemma 10.24, Lemma 10.25 and Lemma 10.29, we thus obtain the result. □

12. Optimal choice of β

Proposition 12.1. *For $\kappa > \frac{1}{2}$ and $\beta > \widetilde{\rho}$, by Proposition 11.8, we can consider*

$$\gamma(\beta) := \sup\{s \in [\beta, +\infty) \mid F^-(s) = 0\}.$$

Then, the minimum of $\gamma(\beta)$ over β is taken when $\beta = \rho+1$ with $\gamma(\rho+1) = \rho+1$.

Proof. To make the dependence of $F^-(s), T^-(s)$ on β visible, write

$$F^-(s) = F^-(s,\beta) \quad \text{and} \quad T^-(s) = T^-(s,\beta)$$

By (10.4), we have

$$s^\kappa F^-(s,\beta) = s^\kappa(1 - T^-(s,\beta)) = \frac{s^\kappa P(s) - s^\kappa Q(s)}{2} = B + A\int_\beta^s \frac{dt^\kappa}{(t-1)^\kappa}.$$

Note that the constants A, B are indeed functions of β given by

$$A(\beta) = \frac{2q(\beta)}{D(\beta)} \quad \text{and} \quad B(\beta) = \left(\frac{\beta-1}{\beta}\right)^{1-\kappa}\frac{2q(\beta-1)}{D(\beta)}$$

as determined in Proposition 11.8. We first evaluate the value of $\gamma(\rho+1)$. Since $q(\rho) = 0$ by definition, we have $B(\rho+1) = 0$. Therefore, we have $F^-(\rho+1, \rho+1) = 0$. Since $s \mapsto F^-(s,\beta)$ is strictly increasing, we find that $\gamma(\rho+1) = \rho+1$. Therefore, by the monotonicity of $s \mapsto F^-(s,\beta)$, it suffices to show

$$F^-(\rho+1,\beta) \le 0 \quad \text{for } \widetilde{\rho} < \beta < \rho+1.$$

Since $\beta < \rho+1 < \beta+1$, we have

$$\begin{aligned} F^-(\rho+1,\beta) \le 0 &\iff B(\beta) + A(\beta)\int_\beta^{\rho+1}\frac{dt^\kappa}{(t-1)^\kappa} \le 0 \\ &\iff \frac{(\beta-1)^{1-\kappa}q(\beta-1)}{\beta^{1-\kappa}q(\beta)} + \int_\beta^{\rho+1}\frac{dt^\kappa}{(t-1)^\kappa} \le 0. \end{aligned}$$

We thus study

$$\varphi(\beta) := \frac{(\beta-1)^{1-\kappa}q(\beta-1)}{\beta^{1-\kappa}q(\beta)} + \int_\beta^{\rho+1}\frac{dt^\kappa}{(t-1)^\kappa}.$$

By (11.2), we have

$$\left(\frac{(\beta-1)^{1-\kappa}q(\beta-1)}{\beta^{1-\kappa}q(\beta)}\right)'$$

$$= \frac{\kappa(\beta-1)^{-\kappa}q(\beta)\cdot\beta^{1-\kappa}q(\beta) - (\beta-1)^{1-\kappa}q(\beta-1)\cdot\kappa\beta^{-\kappa}q(\beta+1)}{\left(\beta^{1-\kappa}q(\beta)\right)^2}$$

$$= \frac{\kappa\beta^{\kappa-1}}{(\beta-1)^{\kappa}} - \frac{\kappa(\beta-1)^{1-\kappa}q(\beta-1)\cdot q(\beta+1)}{\beta^{2-\kappa}q(\beta)^2}$$

and so

$$\varphi'(\beta) = -\frac{\kappa(\beta-1)^{1-\kappa}q(\beta-1)\cdot q(\beta+1)}{\beta^{2-\kappa}q(\beta)^2}.$$

By Proposition 11.2 and the definition of ρ, we then have

$$\varphi'(\beta) > 0 \quad \text{for } \widetilde{\rho} < \beta < \rho+1.$$

This shows

$$\varphi(\beta) < \varphi(\rho+1) = 0 \quad \text{for } \widetilde{\rho} < \beta < \rho+1$$

and completes the proof. □

By Proposition 12.1, we find that the optimal choice of β is

$$\beta = \beta_\kappa = \rho + 1$$

since $\rho_\kappa + 1 = 1$, the least possible β if $\kappa \le \frac{1}{2}$. We therefore take $\beta = \rho_\kappa + 1$ below.

We then check the behavior of β as $\kappa \searrow \frac{1}{2}$.

Proposition 12.2.

(i) *For $\kappa > \frac{1}{2}$, we have $\beta > 1$.*

(ii) *We have $\beta - 1 \sim \pi e^{\gamma}(2\kappa-1)^2$ and so $\beta \to 1$ as $\kappa \searrow \frac{1}{2}$.*

Proof.

(i). When $\kappa > \frac{1}{2}$, by Proposition 10.14, ρ_κ is a genuine zero of the function $q_\kappa(s) = r_{\kappa,\kappa}(s)$ and so $\rho_\kappa > 0$. We then have $\beta_\kappa = \rho_\kappa + 1 > 1$.

(ii). Consider the range $\frac{1}{2} < \kappa < \frac{3}{4}$. We then have $\kappa + \kappa - 1 < 1$. Thus, we can use Lemma 10.5 with $N = 1$ with recalling Lemma 10.7 to obtain

$$q_\kappa(s) = r_{\kappa,\kappa}(s) = s^{2\kappa-1} + \frac{1}{\Gamma(1-2\kappa)}\int_0^\infty \Big(\Phi_\kappa(-x) - 1\Big)e^{-sx}x^{-2\kappa}dx \quad \text{for } s > 0.$$

By substituting $s = \rho_\kappa > 0$ with recalling $q_\kappa(\rho) = 0$, we have

$$\rho_\kappa^{2\kappa-1} = -\frac{1}{\Gamma(1-2\kappa)}\int_0^\infty \Big(\Phi_\kappa(-x) - 1\Big)e^{-\rho_\kappa x}x^{-2\kappa}dx.$$

By changing the variable via $x = \frac{t}{\rho_\kappa}$, we have

$$\rho_\kappa^{\kappa} = -\frac{1}{\Gamma(1-2\kappa)}\int_0^\infty \left(\frac{\Phi_\kappa(-\frac{t}{\rho_\kappa}) - 1}{(\frac{t}{\rho_\kappa})^{\kappa}}\right)e^{-t}t^{-\kappa}dt. \tag{12.1}$$

For $0 < x \le 1$, we have

$$\frac{\Phi_\kappa(-x)-1}{x^\kappa} = \frac{e^{\kappa\,\mathrm{Ein}(x)}-1}{x^\kappa} = \frac{\kappa\,\mathrm{Ein}(x)}{x^\kappa}\sum_{m=0}^{\infty}\frac{(\kappa\,\mathrm{Ein}(x))^m}{(m+1)!} \ll x^{1-\kappa} \ll 1$$

since $\mathrm{Ein}(x) \le \mathrm{Ein}(1)$, where the implicit constant is independent of κ. For $x \ge 1$,

$$\frac{\Phi_\kappa(-x)-1}{x^\kappa} \ll x^{-\kappa}\left(\exp\left(\kappa\int_0^x \frac{1-e^{-t}}{t}dt\right) + 1\right) \ll 1$$

where again the implicit constant is independent of κ. Therefore, we have

$$\int_0^\infty \left(\frac{\Phi_b(-\frac{x}{\rho_\kappa})-1}{(\frac{x}{\rho_\kappa})^\kappa}\right)e^{-x}x^{-\kappa}dx \ll \int_0^\infty e^{-x}(x^{-\frac{3}{4}}+x^{-\frac{1}{2}})dx \ll 1,$$

where the implicit constant is independent of κ. Thus, by (12.1),

$$\rho_\kappa \to 0 \quad \text{as } \kappa \searrow \tfrac{1}{2}.$$

As $x\to\infty$, by Lemma 10.9, we have

$$\begin{aligned}\frac{\Phi_\kappa(-x)}{x^\kappa} &= \exp\left(\kappa\int_0^x \frac{1-e^{-t}}{t}dt - \kappa\int_1^x \frac{dt}{t}\right)\\ &= \exp\left(\kappa\int_0^1 \frac{1-e^{-t}}{t}dt - \kappa\int_1^x \frac{e^{-t}}{t}dt\right)\\ &\to \exp\left(\kappa\int_0^1 \frac{1-e^{-t}}{t}dt - \kappa\int_1^\infty \frac{e^{-t}}{t}dt\right) = e^{\kappa\gamma}.\end{aligned}$$

On inserting this formula and

$$\frac{1}{\Gamma(1-2\kappa)} \sim (1-2\kappa) \quad \text{as } \kappa \searrow \frac{1}{2}$$

into (12.1), we therefore have

$$\rho_\kappa^\kappa \sim \Gamma(1-\kappa)e^{\kappa\gamma}(2\kappa-1) \sim \pi^{\frac{1}{2}}e^{\frac{\gamma}{2}}(2\kappa-1) \quad \text{as } \kappa \searrow \tfrac{1}{2}$$

and so

$$\rho_\kappa \sim \pi^{\frac{1}{2\kappa}}e^{\frac{\gamma}{2\kappa}}(2\kappa-1)^{\frac{1}{\kappa}} \sim \pi e^\gamma(2\kappa-1)^2 \quad \text{as } \kappa \searrow \tfrac{1}{2}$$

since

$$(2\kappa-1)^{\frac{1}{\kappa}-2} = \exp\left(-\frac{1}{\kappa}(2\kappa-1)\log(2\kappa-1)\right) \to 1 \quad \text{as } \kappa \searrow \tfrac{1}{2}.$$

This completes the proof. □

13. The error majorants $\widehat{T}^{\pm}(s)$

Let $\beta=\rho+1$ as chosen in the previous section. In order to estimate the error in the approximation of $T_N(D,z)$, we need two more auxiliary functions

$$\widehat{T}^{\pm}\colon (0,+\infty)\to\mathbb{R}.$$

In order to define these functions, we let

$$\widehat{\kappa} := \begin{cases} \kappa & \text{if } \kappa > \frac{1}{2}, \\ \frac{1}{2}+\delta & \text{if } 0<\kappa\le\frac{1}{2} \end{cases} \tag{13.1}$$

with some small $\delta>0$ and write

$$\widehat{\beta} = \widehat{\beta}_\kappa = \beta_{\widehat{\kappa}}.$$

We may suppose

$$\begin{cases} \widehat{\beta}_\kappa = \beta_\kappa \in (1,+\infty) & \text{if } \kappa>\frac{1}{2}, \\ \widehat{\beta}_\kappa \in (1,2) & \text{if } 0<\kappa\le\frac{1}{2} \end{cases} \tag{13.2}$$

by taking δ sufficiently small by Proposition 12.2.

Then, we require $\widehat{T}^{\pm}(s)$ to satisfy the following conditions.

($\widehat{\mathsf{T}}$1) The functions $\widehat{T}^{\pm}(s)$ are continuous on $(0,+\infty)$.

($\widehat{\mathsf{T}}$2) The functions $\widehat{T}^{\pm}(s)$ are differentiable on $(0, \widehat{\beta}+1) \cup (\widehat{\beta}+1, +\infty)$.
($\widehat{\mathsf{T}}$3) The functions $\widehat{T}^{\pm}(s)$ are solutions of the delay differential equations

$$(s^{\widehat{\kappa}+1}\widehat{T}^{\pm}(s))' = -\widehat{\kappa}s^{\widehat{\kappa}}\widehat{T}^{\mp}(s-1) \quad \text{for } s > \widehat{\beta}+\varepsilon_{\pm}.$$

($\widehat{\mathsf{T}}$4) The initial functions of $\widehat{T}^{\pm}(s)$ are given by

$$\begin{cases} s^{\widehat{\kappa}+1}\widehat{T}^{+}(s) = \widehat{A} & \text{for } 0 < s \le \widehat{\beta}+1, \\ s^{\widehat{\kappa}+1}\widehat{T}^{-}(s) = \widehat{B} & \text{for } 0 < s \le \widehat{\beta}. \end{cases}$$

with some suitable constants $\widehat{A}$ and $\widehat{B}$
($\widehat{\mathsf{T}}$5) The functions $\widehat{T}^{\pm}(s)$ satisfy the decay condition

$$\widehat{T}^{\pm}(s) = O(e^{-s}).$$

In this section, we determine the appropriate values of $\widehat{A}, \widehat{B}$ and prepare some lemmas on these functions $\widehat{T}^{\pm}(s)$. Since ($\widehat{\mathsf{T}}$3) is linear, it is irrelevant to change $\widehat{A}, \widehat{B}$ by multiplying some non-zero constant.

As is done for $T^{\pm}(s)$, we define

$$\begin{cases} \widehat{P}(s) := \widehat{T}^{+}(s) - \widehat{T}^{-}(s) \\ \widehat{Q}(s) := \widehat{T}^{+}(s) + \widehat{T}^{-}(s) \end{cases} \quad \text{for } s \ge \widehat{\beta}.$$

We then have

$$\begin{cases} (s^{\widehat{\kappa}+1}\widehat{P}(s))' := +\widehat{\kappa}s^{\widehat{\kappa}}\widehat{P}(s-1) \\ (s^{\widehat{\kappa}+1}\widehat{Q}(s))' := -\widehat{\kappa}s^{\widehat{\kappa}}\widehat{Q}(s-1) \end{cases} \quad \text{for } s > \widehat{\beta}+1. \tag{13.3}$$

We extend $\widehat{P}, \widehat{Q}$ to $\widehat{\beta}-1 < s \le \widehat{\beta}$ similarly to P, Q. We first observe

$$\begin{cases} s^{\widehat{\kappa}+1}\widehat{P}(s) = \widehat{A} - \widehat{B} + \widehat{A}\displaystyle\int_{\widehat{\beta}}^{s} \frac{\widehat{\kappa}t^{\widehat{\kappa}}}{(t-1)^{\widehat{\kappa}+1}}dt \\ s^{\widehat{\kappa}+1}\widehat{Q}(s) = \widehat{A} + \widehat{B} - \widehat{A}\displaystyle\int_{\widehat{\beta}}^{s} \frac{\widehat{\kappa}t^{\widehat{\kappa}}}{(t-1)^{\widehat{\kappa}+1}}dt \end{cases} \quad \text{for } \widehat{\beta} \le s \le \widehat{\beta}+1. \tag{13.4}$$

Thus, in order to keep (13.3) even for $s \in (\widehat{\beta}, \widehat{\beta}+1)$, we let

$$s^{\widehat{\kappa}+1}\widehat{P}(s) = s^{\widehat{\kappa}+1}\widehat{Q}(s) := \widehat{A} \quad \text{for } 0 < s < \widehat{\beta}. \tag{13.5}$$

By recalling (i) of Proposition 10.8, we next consider the standard solutions

$$\begin{cases} \widehat{p}(s) := r_{\widehat{\kappa}+1,-\widehat{\kappa}}(s) = 1 \\ \widehat{q}(s) := r_{\widehat{\kappa}+1,+\widehat{\kappa}}(s) \end{cases}$$

of the adjoint equation

$$\begin{cases} (s\widehat{p}(s))' := (\widehat{\kappa}+1)\widehat{p}(s) - \widehat{\kappa}\widehat{p}(s+1) \\ (s\widehat{q}(s))' := (\widehat{\kappa}+1)\widehat{q}(s) + \widehat{\kappa}\widehat{q}(s+1). \end{cases} \tag{13.6}$$

associated to (13.3). We next determine the Iwaniec pairings

$$\langle \widehat{P}, \widehat{p} \rangle \quad \text{and} \quad \langle \widehat{Q}, \widehat{q} \rangle.$$

By Lemma 10.2 and (13.4), we have

$$\langle \widehat{P}, \widehat{p} \rangle(s) = -\widehat{B}\widehat{p}(\widehat{\beta})\widehat{\beta}^{-\widehat{\kappa}} + \widehat{A}(\widehat{\beta}-1)^{-\widehat{\kappa}}\widehat{p}(\widehat{\beta}-1) = -\widehat{\beta}^{-\widehat{\kappa}}\widehat{B} + (\widehat{\beta}-1)^{-\widehat{\kappa}}\widehat{A}.$$

since $\widehat{p}(s) = 1$. By ($\widehat{\mathsf{T}}$5) and Lemma 10.26, we should have

$$0 = \langle \widehat{P}, \widehat{p} \rangle(s) = -\widehat{\beta}^{-\widehat{\kappa}} \widehat{B} + (\widehat{\beta} - 1)^{-\widehat{\kappa}} \widehat{A}.$$

We thus take $\widehat{A}, \widehat{B}$ by

$$\widehat{A} := (\widehat{\beta} - 1)^{\widehat{\kappa}} \quad \text{and} \quad \widehat{B} := \widehat{\beta}^{\widehat{\kappa}}. \tag{13.7}$$

which satisfies $\langle \widehat{P}, \widehat{p} \rangle(s) = 0$. Similarly, we have

$$\langle \widehat{Q}, \widehat{q} \rangle(s) = +\widehat{B}\widehat{q}(\widehat{\beta})\widehat{\beta}^{-\widehat{\kappa}} + \widehat{A}(\widehat{\beta} - 1)^{-\widehat{\kappa}} \widehat{q}(\widehat{\beta} - 1) = \widehat{q}(\widehat{\beta}) + \widehat{q}(\widehat{\beta} - 1).$$

By (13.6) with $s = \widehat{\beta} - 1$ and Proposition 10.11, we have

$$\langle \widehat{Q}, \widehat{q} \rangle(s) = \frac{\widehat{\beta} - 1}{\widehat{\kappa}} \widehat{q}'(\widehat{\beta} - 1) = \frac{\widehat{\beta} - 1}{\widehat{\kappa}} r'_{\widehat{\kappa}+1,\widehat{\kappa}}(\widehat{\beta} - 1) = 2(\widehat{\beta} - 1) r_{\widehat{\kappa},\widehat{\kappa}}(\widehat{\beta} - 1).$$

By recalling the definition of $q_\kappa, \beta_\kappa, \rho_\kappa$, we have

$$\langle \widehat{Q}, \widehat{q} \rangle(s) = 2(\widehat{\beta} - 1) q_{\widehat{\kappa}}(\rho_{\widehat{\kappa}}) = 0$$

since $\rho_{\widehat{\kappa}}$ is, by definition, the largest zero of $q_{\widehat{\kappa}}(s)$. In summary, we get

$$\langle \widehat{P}, \widehat{p} \rangle(s) = \langle \widehat{Q}, \widehat{q} \rangle(s) = 0. \tag{13.8}$$

We have

$$|\widehat{P}(s)| = \frac{(\widehat{\beta} - 1)^{\widehat{\kappa}}}{s^{\widehat{\kappa}+1}} = \widehat{Q}(s) \quad \text{for } \widehat{\beta} - 1 < s < \widehat{\beta}. \tag{13.9}$$

and $\widehat{Q}(s)$ is not constantly zero for $\widehat{\beta} - 1 < s < \widehat{\beta}$.

Then, we have the following basic properties for $\widehat{T}^{\pm}(s)$:

Proposition 13.1. *For the functions $\widehat{T}^{\pm}(s)$ defined as above, we have*

(i) *For $s > 0$, we have*

$$\widehat{T}^{\pm}(s) > 0 \quad \textit{and} \quad \widehat{T}^{+}(s) \asymp \widehat{T}^{-}(s) \asymp \widehat{Q}(s),$$

where the implicit constant depends only on κ and δ.

(ii) *For $s \geq 1$, we have*

$$\widehat{T}^{\pm}(s) \ll \exp(-s \log s - s \log\log 3s + s \log e\widehat{\kappa})$$

and

$$\widehat{T}^{\pm}(s) = \exp\left(-s \log s - s \log\log 3s + s \log e\widehat{\kappa} + O\left(\frac{s \log\log 3s}{\log 2s}\right)\right),$$

where the implicit constant depends only on κ and δ.

(iii) *For $s > 1$, we have*

$$(s-1)^{\widehat{\kappa}+1} \widehat{T}^{\pm}(s-1) \asymp s^{\widehat{\kappa}+1} \widehat{T}^{\pm}(s)(s \log 3s),$$

where the implicit constant depends only on κ and δ.

Proof.

(i) By ($\widehat{\mathsf{T}}$4), (13.5) and (13.7), the assertion is trivial for $0 < s < \widehat{\beta}$. It thus suffices to consideer the range $s \geq \widehat{\beta}$. We first prove that

$$\widehat{q}(s) > 0 \quad \text{for } s \geq \widehat{\beta}. \tag{13.10}$$

As we saw above, Proposition 10.11 gives

$$\widehat{q}'(s) = r'_{\widehat{\kappa}+1,\widehat{\kappa}}(s) = 2\widehat{\kappa} r_{\widehat{\kappa},\widehat{\kappa}}(s) = 2\widehat{\kappa} q_{\widehat{\kappa}}(s).$$

By the definition of $\widehat{\beta}$, we have

$$\widehat{q}'(s) > 0 \quad \text{for } s > \widehat{\beta} - 1.$$

By (13.6), we thus have

$$0 \le \frac{(s-1)\widehat{q}'(s-1)}{\widehat{\kappa}} = \widehat{q}(s-1) + \widehat{q}(s) < \widehat{q}(s) \quad \text{for } s \ge \widehat{\beta}$$

as desired. Therefore, by (13.8) and (13.9), we can use Lemma 10.17 to conclude that there is $\eta = \eta(\kappa) \in (0,1)$ such that

$$|\widehat{P}(s)| < \eta \widehat{Q}(s) \quad \text{for } s \ge \widehat{\beta}.$$

We then have

$$\widehat{T}^{\pm}(s) = \frac{\pm\widehat{P}(s) + \widehat{Q}(s)}{2} \le \frac{1}{2}(\widehat{Q}(s) + |\widehat{P}(s)|) \ge \frac{1+\eta}{2}\widehat{Q}(s) \quad \text{for } s \ge \widehat{\beta}$$

and

$$\widehat{T}^{\pm}(s) = \frac{\pm\widehat{P}(s) + \widehat{Q}(s)}{2} \ge \frac{1}{2}(\widehat{Q}(s) - |\widehat{P}(s)|) \ge \frac{1-\eta}{2}\widehat{Q}(s) > 0 \quad \text{for } s \ge \widehat{\beta}$$

and so

$$\widehat{T}^{\pm}(s) \asymp \widehat{Q}(s).$$

This completes the proof.

(ii)., (iii) By (13.5), $\widehat{Q}(s) > 0$ for $\widehat{\beta} - 1 < s < \widehat{\beta}$. Hence, in the range $s \ge \widehat{\beta}$, (ii) and (iii) follows by (i) proven above, Lemma 10.24, Lemma 10.25 and Lemma 10.29 with using (13.8) and (13.10). For the remaining range $1 \le s \le \widehat{\beta}$, the assertion (ii) is trivial. For the remaining range $0 < s \le \widehat{\beta}$, the assertion (iii) follows by the definition (13.5) of the extended part of $\widehat{Q}(s)$. □

We also need the following two inequalities:

Lemma 13.2. *We have*

$$s^{\kappa} T^{\pm}(s) \ll s^{\widehat{\kappa}+1}\widehat{T}^{\pm}(s) \quad \text{for } s \in I_{\pm},$$

where the implicit constant depends only on κ and δ.

Proof. When $\beta - 1 < s \le \beta + 1 \le \widehat{\beta} + 1$, we have

$$s^{\kappa} T^{+}(s) = A - s^{\kappa} \quad \text{and} \quad s^{\widehat{\kappa}+1}\widehat{T}^{+}(s) = \widehat{A}$$

as in (10.5) and (13.5). Since $A, \widehat{A} > 0$ and $A > (\beta+1)^{\kappa}$, we have

$$0 \le s^{\kappa} T^{+}(s) \le \frac{A}{\widehat{A}} \cdot s^{\widehat{\kappa}+1}\widehat{T}^{+}(s) \quad \text{for } \beta - 1 < s \le \beta + 1.$$

By (iv) of Proposition 11.8 and (i) of Proposition 13.1, we also have

$$s^{\kappa} T^{\pm}(s) \ll s^{\widehat{\kappa}+1}\widehat{T}^{\pm}(s) \quad \text{for } \beta \le s \le \beta + 2.$$

By the positivity of $f_n(s)$, we thus have

$$s^{\kappa} T^{\pm}(s) \le C \cdot s^{\widehat{\kappa}+1}\widehat{T}^{\pm}(s) \quad \text{for } s \in (\beta - 1, \beta + 2] \cap I_N. \tag{13.11}$$

with some real number $C = C(\kappa) \geq 1$. By (ii) of Proposition 13.1 and ($\widehat{\mathsf{T}}$3),

$$s^{\widehat{\kappa}+1}\widehat{T}^{\pm}(s) = \widehat{\kappa}\int_s^{\infty} t^{\widehat{\kappa}}\widehat{T}^{\mp}(t-1)dt \quad \text{for } s \geq \widehat{\beta}. \tag{13.12}$$

By the convergence part of Proposition 11.8, it thus suffices to prove

$$s^{\kappa}T_N(s) \leq C \cdot s^{\widehat{\kappa}+1}\widehat{T}^{(-)^N}(s) \quad \text{for } s \in I_N \text{ and } N \geq 1, \tag{13.13}$$

where C is the same one as in (13.11). We use induction on $N \geq 1$.

Initial case $N = 1$. Recall that $T_1(s)$ is supported only on $(\beta-1, \beta+1]$ as stated in Proposition 9.3. Therefore, (13.13) follows by (13.11).

Induction step from $N-1$ to N. By (13.11), we may assume $s \geq \beta+2$. By (ii) and (iii) of Proposition 9.3 and Lemma 11.10, by integrating, we have

$$s^{\kappa}T_N(s) = \kappa\int_s^{\infty} t^{\kappa-1}T_{N-1}(t-1)dt \quad \text{for } s \geq \beta+1.$$

By (13.2), we have

$$s \geq \beta + 2 \geq \widehat{\beta} + 1$$

and so by the induction hypothesis and (13.12), we then have

$$\begin{aligned} s^{\kappa}T_N(s) &\leq \kappa\int_s^{\infty} t^{\kappa-1}T_{N-1}(t-1)dt \\ &\leq C\cdot\kappa\int_s^{\infty} t^{\kappa-1}(t-1)^{\widehat{\kappa}-\kappa+1}\widehat{T}^{(-)^{N-1}}(t-1)dt \\ &\leq C\cdot\widehat{\kappa}\int_s^{\infty} t^{\widehat{\kappa}}\widehat{T}^{(-)^{N-1}}(t-1)dt \leq C\cdot s^{\widehat{\kappa}+1}\widehat{T}^{(-)^N}(s) \end{aligned}$$

This proves (13.13) for the N-th case. □

Lemma 13.3. *For a real number θ satisfying*

$$\theta \in \begin{cases} (0,+\infty) & \text{if } \kappa > \frac{1}{2}, \\ (\frac{1}{2},+\infty) & \text{if } 0 < \kappa \leq \frac{1}{2}, \end{cases}$$

we have

$$s^{\widehat{\kappa}+1}\widehat{T}^{\pm}(s) > \widehat{\kappa}\int_s^{\infty}\left(\frac{t-1}{t}\right)^{\theta} t^{\widehat{\kappa}}\widehat{T}^{\mp}(t-1)dt \quad \text{for } s \geq \beta + \varepsilon_{\pm}$$

provided δ is sufficiently small in terms of θ and κ, i.e. $0 < \delta \leq \delta_0(\theta,\kappa)$.

Proof. By (ii) of Proposition 13.1 and the defining equation of $\widehat{T}^{\pm}(s)$, we have

$$s^{\widehat{\kappa}+1}\widehat{T}^{\pm}(s) = \widehat{\kappa}\int_s^{\infty} t^{\widehat{\kappa}}\widehat{T}^{\mp}(t-1)dt \quad \text{for } s \geq \widehat{\beta} + \varepsilon_{\pm}. \tag{13.14}$$

Thus, the assertion is obvious for $s \geq \widehat{\beta} + \varepsilon_{\pm}$ since

$$\left(\frac{t-1}{t}\right)^{\theta} < 1.$$

We thus consider the case $\beta+\varepsilon_\pm \le s < \widehat{\beta}+\varepsilon_\pm$, which occurs only if $0<\kappa\le\frac{1}{2}$. In this case, we have $\beta=1$, $1<\widehat{\beta}<2$ and $\theta>\frac{1}{2}$. Also, in this range $\beta+\varepsilon_\pm \le s < \widehat{\beta}+\varepsilon_\pm$, as defined in ($\widehat{\mathsf{T}}$4), we have

$$s^{\widehat{\kappa}+1}\widehat{T}^{\pm}(s)=\widehat{\beta}^{\widehat{\kappa}+1}\widehat{T}^{\pm}(\widehat{\beta}).$$

Thus, it suffices to show

$$\widehat{\beta}^{\widehat{\kappa}+1}\widehat{T}^{\pm}(\widehat{\beta})>\widehat{\kappa}\int_{1+\varepsilon_\pm}^{\infty}\left(\frac{t-1}{t}\right)^{\theta}t^{\widehat{\kappa}}\widehat{T}^{\mp}(t-1)dt.$$

By (13.14), we have

$$\widehat{\beta}^{\widehat{\kappa}+1}\widehat{T}^{\pm}(\widehat{\beta})=(\widehat{\beta}+\varepsilon_\pm)^{\widehat{\kappa}+1}\widehat{T}^{\pm}(\widehat{\beta}+\varepsilon_\pm)=\widehat{\kappa}\int_{\widehat{\beta}+\varepsilon_\pm}^{\infty}t^{\widehat{\kappa}}\widehat{T}^{\mp}(t-1)dt.$$

We therefore have

$$\widehat{\beta}^{\widehat{\kappa}+1}\widehat{T}^{\pm}(\widehat{\beta})-\widehat{\kappa}\int_{1+\varepsilon_\pm}^{\infty}\left(\frac{t-1}{t}\right)^{\theta}t^{\widehat{\kappa}}\widehat{T}^{\mp}(t-1)dt=\widehat{\kappa}(I^{\pm}-J^{\pm}),$$

where

$$I^{\pm}:=\int_{\widehat{\beta}+\varepsilon_\pm}^{\infty}\left(1-\left(\frac{t-1}{t}\right)^{\theta}\right)t^{\widehat{\kappa}}\widehat{T}^{\mp}(t-1)dt$$
$$J^{\pm}:=\int_{1+\varepsilon_\pm}^{\widehat{\beta}+\varepsilon_\pm}\left(\frac{t-1}{t}\right)^{\theta}t^{\widehat{\kappa}}\widehat{T}^{\mp}(t-1)dt.$$

Therefore, it suffices to prove

(13.15) $$I^{\pm}>J^{\pm}\quad\text{for sufficiently small } \delta>0.$$

We now consider the signs $\pm$ separately.

Case I. The sign $+$. By ($\widehat{\mathsf{T}}$3) and (i) of Proposition 13.1, $\widehat{T}^{-}(s)$ is decreasing for $s\ge\widehat{\beta}$. Therefore, by recalling $\theta>\frac{1}{2}$, we have

$$\begin{aligned}I^{+}&\ge\int_{\widehat{\beta}+1}^{\widehat{\beta}+2}\left(1-\left(\frac{t-1}{t}\right)^{\theta}\right)t^{\widehat{\kappa}}\widehat{T}^{-}(t-1)dt\\&\ge\left(1-\left(\frac{\widehat{\beta}+1}{\widehat{\beta}+2}\right)^{\theta}\right)\widehat{T}^{-}(\widehat{\beta}+1)\\&\ge(1-(\tfrac{3}{4})^{\frac{1}{2}})\widehat{T}^{-}(\widehat{\beta}+1).\end{aligned}$$

By ($\widehat{\mathsf{T}}$3) and ($\widehat{\mathsf{T}}$4), we have

$$s^{\widehat{\kappa}+1}\widehat{T}^{-}(s)=\widehat{B}-\widehat{A}\int_{\widehat{\beta}}^{s}\frac{\widehat{\kappa}t^{\widehat{\kappa}}}{(t-1)^{\widehat{\kappa}+1}}dt\quad\text{for } \widehat{\beta}\le s\le\widehat{\beta}+2.$$

By substituting $s=\widehat{\beta}+1$ and $s=\widehat{\beta}+2$ and then taking the difference, we have

$$\begin{aligned}(\widehat{\beta}+1)^{\widehat{\kappa}+1}\widehat{T}^{-}(\widehat{\beta}+1)&>(\widehat{\beta}+1)^{\widehat{\kappa}+1}\widehat{T}^{-}(\widehat{\beta}+1)-(\widehat{\beta}+2)^{\widehat{\kappa}+1}\widehat{T}^{-}(\widehat{\beta}+2)\\&=\widehat{A}\int_{\widehat{\beta}+1}^{\widehat{\beta}+2}\frac{\widehat{\kappa}t^{\widehat{\kappa}}}{(t-1)^{\widehat{\kappa}+1}}dt\ge\frac{\widehat{\kappa}}{\widehat{\beta}+1}\widehat{A}=\frac{\widehat{\kappa}}{\widehat{\beta}+1}(\widehat{\beta}-1)^{\widehat{\kappa}}\end{aligned}$$

and so

$$I^+ \geq (1-(\tfrac{3}{4})^{\frac{1}{2}})\widehat{T}^-(\widehat{\beta}+1) \geq (1-(\tfrac{3}{4})^{\frac{1}{2}})\frac{\widehat{\kappa}}{\widehat{\beta}+1}(\widehat{\beta}-1)^{\widehat{\kappa}}$$

On the other hand, by ($\widehat{\mathsf{T}}$4), by choosing δ small so that $\theta > \widehat{\kappa}$, we have

$$\begin{aligned} J^+ &= \int_2^{\widehat{\beta}+1}\left(\frac{t-1}{t}\right)^{\theta} t^{\widehat{\kappa}}\widehat{T}^-(t-1)dt \\ &= \widehat{B}\int_2^{\widehat{\beta}+1} t^{\widehat{\kappa}-\theta}(t-1)^{\theta-\widehat{\kappa}-1}dt \leq \widehat{\beta}^{\theta}\widehat{B}\int_2^{\widehat{\beta}+1} dt = \widehat{\beta}^{\widehat{\kappa}+\theta}(\widehat{\beta}-1). \end{aligned}$$

By recalling the definition of $\widehat{\beta}$, we have

$$\widehat{\beta} = \beta_{\widehat{\kappa}} = \beta_{\kappa+\delta} \to 1 \quad \text{as } \delta \searrow 0$$

as proved in Proposition 12.2. Therefore, we have

$$I^+ \gg (\widehat{\beta}-1)^{\widehat{\kappa}}, \quad J^+ \ll (\widehat{\beta}-1) \quad \text{and} \quad \widehat{\kappa} < 1 \quad \text{as } \delta \searrow 0$$

and (13.15) holds when $\pm = +$.

Case II. The sign $-$. By ($\widehat{\mathsf{T}}$4) and (13.7), we have

$$\begin{aligned} I^- &\geq \int_{\widehat{\beta}}^{\widehat{\beta}+1}\left(1-\left(\frac{t-1}{t}\right)^{\theta}\right)t^{\widehat{\kappa}}\widehat{T}^+(t-1)dt \\ &\geq \left(1-\left(\frac{\widehat{\beta}}{\widehat{\beta}+1}\right)^{\theta}\right)\int_{\widehat{\beta}}^{\widehat{\beta}+1}\widehat{T}^+(t-1)dt \\ &\geq (1-(\tfrac{2}{3})^{\frac{1}{2}})\int_{\widehat{\beta}}^{\widehat{\beta}+1}\widehat{T}^+(t-1)dt \\ &= (1-(\tfrac{2}{3})^{\frac{1}{2}})\widehat{A}\int_{\widehat{\beta}}^{\widehat{\beta}+1}(t-1)^{-(\widehat{\kappa}+1)}dt \\ &= (1-(\tfrac{2}{3})^{\frac{1}{2}})\frac{\widehat{A}}{\widehat{\kappa}}\Big((\widehat{\beta}-1)^{-\widehat{\kappa}}-\widehat{\beta}^{-\widehat{\kappa}}\Big) \\ &= (1-(\tfrac{2}{3})^{\frac{1}{2}})\frac{1}{\widehat{\kappa}}\Big(1-\widehat{\beta}^{-\widehat{\kappa}}(\widehat{\beta}-1)^{\widehat{\kappa}}\Big) \end{aligned}$$

By ($\widehat{\mathsf{T}}$4) and (13.7) again, and taking δ small enough to make

$$\widehat{\kappa} = \tfrac{1}{2}+\delta < \theta - \tfrac{1}{2}(\theta-\tfrac{1}{2})$$

(recall that $\theta > \frac{1}{2}$ and so $\theta - \frac{1}{2}(\theta-\frac{1}{2}) > \frac{1}{2}$), we also have

$$\begin{aligned} J^- = \int_1^{\widehat{\beta}}\left(\frac{t-1}{t}\right)^{\theta}t^{\widehat{\kappa}}\widehat{T}^+(t-1)dt &= \widehat{A}\int_1^{\widehat{\beta}} t^{\widehat{\kappa}-\theta}(t-1)^{\theta-\widehat{\kappa}-1}dt \\ &\leq \widehat{A}\int_1^{\widehat{\beta}}(t-1)^{\frac{1}{2}(\theta-\frac{1}{2})-1}dt \\ &= \frac{2\widehat{A}}{(\theta-\frac{1}{2})}(\widehat{\beta}-1)^{\frac{1}{2}(\theta-\frac{1}{2})} \\ &= \frac{2}{(\theta-\frac{1}{2})}(\widehat{\beta}-1)^{\widehat{\kappa}+\frac{1}{2}(\theta-\frac{1}{2})} \end{aligned}$$

We thus have

$$J^- \to 0 \quad \text{as } \delta \searrow 0.$$

By recalling the definition of $\widehat{\beta}$, we have

$$\widehat{\beta} = \beta_{\widehat{\kappa}} = \beta_{\kappa+\delta} \to 1 \quad \text{as } \delta \searrow 0$$

as proved in Proposition 12.2. Therefore, we have

$$I^- \to (1 - (\tfrac{2}{3})^{\frac{1}{2}})\frac{1}{\widehat{\kappa}} > 0 \quad \text{and} \quad J^- \to 0 \quad \text{as } \delta \searrow 0$$

and (13.15) holds when $\pm = -$. □

14. Completion of the proof of the Rosser–Iwaniec sieve

We now move on to the completion of the proof of the Rosser–Iwaniec sieve. The remaining task is to estimate the sum

$$V_n(D,z) = \sum_{\substack{z > p_1 > \cdots > p_n \\ p_1 \cdots p_m p_m^\beta < D \ (1 \le m < n,\, m \equiv n \ (\mathrm{mod}\ 2)) \\ p_1 \cdots p_n p_n^\beta \ge D}} \omega(p_1 \cdots p_n) V(p_n).$$

defined in (6.2). Recall that

$$V_n(D,z) = \begin{cases} \displaystyle\sum_{y_1 \le p < z} \omega(p) V(p) & \text{if } n = 1, \\ \displaystyle\sum_{y_n \le p < z_n} \omega(p) V_{n-1}\left(\frac{D}{p}, p\right) & \text{if } n \ge 2 \text{ and } s \ge \beta - \varepsilon_n \end{cases}$$

as proved in Lemma 7.1. Let us introduce

$$T_N(D,z) \coloneqq \sum_{\substack{1 \le n \le N \\ n \equiv N \ (\mathrm{mod}\ 2)}} V_n(D,z).$$

Lemma 14.1. *For $D, z \ge 2$, $n \in \mathbb{N}$ and $\omega \in \Omega(\kappa, K)$, we have*

$$V_n(D,z) \le \frac{\mathcal{L}^n}{n!}$$

where

$$\mathcal{L} = \mathcal{L}(z,\kappa,K) \coloneqq \kappa \log \frac{\log z}{\log 2} + \log\left(1 + \frac{K}{\log 2}\right).$$

Proof. We have

$$\begin{aligned} V_n(D,z) &\le \sum_{z > p_1 > \cdots > p_n} \omega(p_1 \cdots p_n) V(p_n) \\ &\le \sum_{z > p_1 > \cdots > p_n} \omega(p_1 \cdots p_n) \\ &\le \frac{1}{n!}\Big(\sum_{p<z} \omega(p)\Big)^n. \end{aligned}$$

Since $\omega \in \Omega(\kappa, K)$, we have

$$\sum_{p<z} \omega(p) \le \sum_{p<z} \log(1-\omega(p))^{-1} = \log \frac{V(2)}{V(z)} \le \kappa \log \frac{\log z}{\log 2} + \log\left(1 + \frac{K}{\log 2}\right)$$

and so the assertion follows. $\square$

Lemma 14.2. *For $x \ge 0$ and $N \in \mathbb{Z}_{\ge 0}$, we have*

$$\frac{x^N}{N!} \le \sum_{n=N}^{\infty} \frac{x^n}{n!} \le \frac{x^N}{N!} e^x.$$

Proof. The first bound is obvious. The second bound can be obtained by

$$\sum_{n=N}^{\infty} \frac{x^n}{n!} = \sum_{m=0}^{\infty} \frac{x^{N+m}}{(N+m)!} = \frac{x^N}{N!} \sum_{m=0}^{\infty} \frac{1}{\binom{N+m}{m}} \frac{x^m}{m!} \le \frac{x^N}{N!} \sum_{m=0}^{\infty} \frac{x^m}{m!} \le \frac{x^N}{N!} e^x.$$

This completes the proof. $\square$

Lemma 14.3. *For $D \ge z \ge 2$ with $s \ge \beta$, $N \in \mathbb{N}$ and $\omega \in \Omega(\kappa, K)$, we have*

$$T_N(D,z) \le \exp\big(-s \log s + s \log \mathcal{L} + s + \mathcal{L} + O(\log 2s)\big),$$

where the implicit constants depends at most on κ.

Proof. Since we have

$$s \ge \beta + n \implies V_n(D,z) = 0,$$

we have

$$T_N(D,z) = \sum_{\substack{1 \le n \le N \\ n \equiv N \ (\mathrm{mod}\ 2)}} V_n(D,z) = \sum_{\substack{s-\beta < n \le N \\ n \equiv N \ (\mathrm{mod}\ 2)}} V_n(D,z).$$

By Lemma 14.1, we have

$$T_N(D,z) \le \sum_{n > s-\beta} \frac{\mathcal{L}^n}{n!} = \sum_{n=[s-\beta]+1}^{\infty} \frac{\mathcal{L}^n}{n!}.$$

Since $\mathcal{L} \ge 0$, by Lemma 14.2, we have

$$T_N(D,z) \le \frac{\mathcal{L}^{[s-\beta]+1}}{([s-\beta]+1)!} e^{\mathcal{L}}.$$

By using the bound

$$n! \ge \exp(n \log n - n)$$

obtained by

$$e^n = \sum_{m=0}^{\infty} \frac{n^n}{n!} \ge \frac{n^n}{n!} \quad \text{and so} \quad n! \ge \left(\frac{n}{e}\right)^n = \exp(n \log n - n)$$

and the bound $[s-\beta]+1 \le s-\beta+1 \le s$, we have

$$\begin{aligned} &T_N(D,z) \\ &\le \exp\Big(-([s-\beta]+1)\log([s-\beta]+1) + ([s-\beta]+1) + ([s-\beta]+1)\log \mathcal{L} + \mathcal{L}\Big) \\ &\le \exp\Big(-([s-\beta]+1)\log([s-\beta]+1) + s\log \mathcal{L} + s + \mathcal{L}\Big). \end{aligned}$$

If $\beta \le s < 2\beta$, we trivially have

$$-([s-\beta]+1)\log([s-\beta]+1) = -s\log s + O(\log 2s)$$

If $s \ge 2\beta$, since $[s-\beta]+1 > s-\beta \ge \beta \ge 1$, we have

$$-([s-\beta]+1)\log([s-\beta]+1) \le -(s-\beta)\log(s-\beta) = -s\log s + O(\log 2s).$$

Therefore, we have

$$T_N(D,z) \le \exp\bigl(-s\log s + s\log\mathcal{L} + s + \mathcal{L} + O(\log 2s)\bigr).$$

This completes the proof. □

Lemma 14.4. *For $N \in \mathbb{N}$, $D, z \ge 2$ with $s := \frac{\log D}{\log z} \in I_N$, $\Delta \in (0, \Delta_0)$ with*

$$\Delta_0 = \Delta_0(\kappa) = \begin{cases} 1 & \text{if } \kappa > \frac{1}{2}, \\ \frac{1}{2} & \text{if } 0 < \kappa \le \frac{1}{2} \end{cases}$$

and a real number d with

$$d > \frac{7}{\Delta_0 - \Delta} \tag{14.1}$$

and $\omega \in \Omega(\kappa, K)$, we have

$$T_N(D,z) \le V(z)\bigl(T_N(s) + Ce^{\sqrt{K}}E_N(D,s)(\log D)^{-\Delta}\bigr),$$

where the function $E_N(D,s)$ is defined by

$$E_N(D,s) := \left(1 + \frac{s^d}{\log D}\right)^s s^{\widehat{\kappa}-\kappa+1}\widehat{T}^{\pm}(s)$$

with

$$\pm := \begin{cases} + & \text{if } N \text{ is odd}, \\ - & \text{if } N \text{ is even} \end{cases}$$

and the constants $C \ge 1$ depends only on κ, Δ, d.

Proof. We choose δ in (13.1) based on κ and Δ so that Lemma 13.3 with

$$\theta := 1 - \Delta_0 + \tfrac{1}{2}(\Delta_0 - \Delta) \in \begin{cases} (0, +\infty) & \text{if } \kappa > \frac{1}{2}, \\ (\frac{1}{2}, +\infty) & \text{if } 0 < \kappa \le \frac{1}{2}, \end{cases}$$

i.e. the inequality

$$\widehat{\kappa}\int_s^\infty \left(\frac{t-1}{t}\right)^{1-\Delta_0+\frac{1}{2}(\Delta_0-\Delta)} t^{\widehat{\kappa}}\widehat{T}^{\mp}(t-1)dt \le s^{\widehat{\kappa}+1}\widehat{T}^{\pm}(s) \tag{14.2}$$

holds for $s \ge \beta + \varepsilon_\pm$.

We also take $\Theta_K > 0$ fixed for a given κ, Δ, d such that

$$\frac{1}{\Theta_K} > \frac{2}{d}, \tag{14.3}$$

$$\frac{2}{\Theta_K} + \frac{3}{d} < \Delta_0 - \Delta \tag{14.4}$$

which is possible since

$$\frac{4}{d} + \frac{3}{d} < \Delta_0 - \Delta$$

by the assumption (14.1)

We next show a preliminary estimate. Indeed, this estimate proves the assertion when D is small or when s is large. Let

$$\sigma := (\log D)^{\frac{1}{d}}(\log\log 27D).$$

For later necessity, we prove an estimate bit stronger than the lemma.

Claim 14.5. *When $s \geq \beta$ and*

$$\log D \leq C_1 K^{\Theta_K} \quad \text{or} \quad s \geq \sigma, \tag{14.5}$$

we have

$$T_N(D,z) \ll V(D) \cdot \frac{1}{\log D}\frac{e^{\sqrt{K}}}{\sigma}E_N(D,s)(\log D)^{-\Delta},$$

where the implicit constant is independent of C_1 if $\log D \geq C_1 K^{\Theta_K}$.

Proof. Since $\omega \in \Omega(\kappa, K)$, we have

$$V(D)^{-1} \ll K(\log D)^{\kappa}.$$

Thus, by (ii) of Proposition 13.1, we have

$$\begin{aligned} &V(D) \cdot \frac{1}{\log D}\frac{e^{\sqrt{K}}}{\sigma}E_N(D,s)(\log D)^{-\Delta} \\ &= V(D) \cdot \frac{1}{\log D}\frac{e^{\sqrt{K}}}{\sigma}\left(1 + \frac{s^d}{\log D}\right)^s s^{\widehat{\kappa}-\kappa+1}\widehat{T}^{\pm}(s)(\log D)^{-\Delta} \\ &\gg \exp\biggl(-s\log s - s\log\log 3s + s\log\left(1 + \frac{s^d}{\log D}\right) + \sqrt{K} \\ &\qquad\qquad + O\Bigl(\log K + \log\log D + s\Bigr)\biggr) \end{aligned} \tag{14.6}$$

since we have $s \geq \beta$ by the assumption. We consider two cases separately.

Case A. $\log D \leq C_1 K^{\Theta_K}$. We use Lemma 14.3. We have

$$\begin{aligned} \mathscr{L} &= \kappa \log \frac{\log z}{\log 2} + \log\left(1 + \frac{K}{\log 2}\right) \\ &\ll \log\log D + \log K \ll \log K. \end{aligned} \tag{14.7}$$

When $s \leq K^{\frac{1}{2}}(\log K)^{-1}$, by Lemma 14.3 and (14.6), we have

$$\begin{aligned} T_N(D,z) &\leq \exp\bigl(-s\log s + s\log\log 3K + O(\log K + s)\bigr) \\ &\ll \exp\bigl(-s\log s - s\log\log 3s + \tfrac{1}{2}\sqrt{K}\bigr) \\ &\ll V(D) \cdot \frac{1}{\log D}\frac{e^{\sqrt{K}}}{\sigma}E_N(D,s)(\log D)^{-\Delta}. \end{aligned}$$

When $s \geq K^{\frac{1}{2}}(\log K)^{-1}$, we have

$$\log D \ll K^{\Theta_K} \ll s^{2\Theta_K}(\log s)^{2\Theta_K}$$

and so

$$s\log\left(1 + \frac{s^d}{\log D}\right) \geq s\log\log 3K + 2s\log\log 3s + O(1)$$

by using

$$d - 2\Theta_K > 0$$

assured by (14.3). Thus, by Lemma 14.3 and (14.6), we have

$$\begin{aligned} T_N(D,z) &\le \exp\big(-s\log s + s\log\log 3K + O(\log K + s)\big) \\ &\ll \exp\left(-s\log s - 2s\log\log 3s + s\log\left(1 + \frac{s^d}{\log D}\right) + \tfrac{1}{2}\sqrt{K} + O(s)\right) \\ &\ll V(D)\cdot\frac{1}{\log D}\frac{e^{\sqrt{K}}}{\sigma}E_N(D,s)(\log D)^{-\Delta}. \end{aligned}$$

Therefore, the claim holds.

Case B. $\log D \ge C_1 K^{\Theta_K}$ and $s \ge \sigma$. In this case, by (14.7), we have

$$\mathscr{L} \ll (\log\log 27D)\left(1 + \frac{\log K}{\log\log 27D}\right) \ll \log\log 27D.$$

Also, since $s \ge \sigma = (\log D)^{\frac{1}{d}}(\log\log 27D)$, we have

$$s \ge (\log D)^{\frac{1}{d}} \quad \text{and} \quad \frac{s^d}{\log D} \gg (\log 3s)^d.$$

We thus have

$$\log\mathscr{L} \le \log\log\log 3D + O(1) \le \log\log 3s + O(1),$$

$$s\log\left(1 + \frac{s^d}{\log D}\right) \ge ds\log\log 3s + O(s).$$

We also have

$$\mathscr{L} \ll \log\log 27D \ll \log s.$$

Therefore, by Lemma 14.3 and (14.6), we have

$$\begin{aligned} T_N(D,z) &\le \exp\Big(-s\log s + s\log\log 3s + O(s)\Big) \\ &\le \exp\left(-s\log s - (d-1)s\log\log 3s + s\log\left(1 + \frac{s}{\log D}\right) + O(s)\right) \\ &\ll V(D)\cdot\frac{1}{\log D}\frac{e^{\sqrt{K}}}{\sigma}E_N(D,s)(\log D)^{-\Delta} \end{aligned}$$

since (14.1) implies

$$d > \frac{7}{\Delta_0 - \Delta} > 7.$$

Therefore, the claim holds even for this case. □

We now prove the lemma by induction on N.

Initial case $N = 1$. For $T_1(D,z)$, we have

$$T_1(D,z) = V_1(D,z) = 0 \quad \text{for } s > \beta + 1.$$

Thus, we may assume $\beta - 1 < s \le \beta + 1$. By Lemma 7.1 and Lemma 8.4, recalling

$$y_1 = D^{\frac{1}{\beta+1}} \quad \text{and} \quad z = D^{\frac{1}{s}} \ge y_1,$$

we have

$$\begin{aligned}T_1(D,z)=V_1(D,z)&=\sum_{y_1\le p<z}\omega(p)V(p)\\&=\sum_{\max(y_1,2)\le p<z}\omega(p)V(p)\\&=V(z)\sum_{\max(y_1,2)\le p<z}\omega(p)\frac{V(p)}{V(z)}\\&=V(z)\biggl(\frac{V(\max(y_1,2))}{V(z)}-1\biggr).\end{aligned}$$

Since $\omega\in\Omega(\kappa,K)$ and $z\ge\max(y_1,2)\ge2$, we have

$$\begin{aligned}T_1(D,z)&\le V(z)\biggl(\biggl(\frac{\log z}{\log\max(y_1,2)}\biggr)^{\kappa}-1+\frac{K}{\log z}\biggl(\frac{\log z}{\log\max(y_1,2)}\biggr)^{\kappa+1}\biggr)\\&\le V(z)\biggl(\biggl(\frac{\log z}{\log y_1}\biggr)^{\kappa}-1+\frac{K}{\log z}\biggl(\frac{\log z}{\log y_1}\biggr)^{\kappa+1}\biggr)\\&\le V(z)\biggl(\frac{(\beta+1)^{\kappa}-s^{\kappa}}{s^{\kappa}}+\frac{K(\beta+1)^{\kappa+1}}{s^{\kappa}\log D}\biggr)\\&=V(z)\biggl(T_1(s)+\frac{K(\beta+1)^{\kappa+1}}{s^{\kappa}\log D}\biggr).\end{aligned}$$

For $\beta-1<s<\beta+1$, by ($\widehat{\mathsf{T}}$4) and (13.7), we have

$$(\beta+1)^{\kappa+1}s^{-\kappa}\ll s^{\widehat{\kappa}-\kappa+1}\widehat{T}^{+}(s).$$

Therefore, for large $C\ge C(\kappa,\Delta)$, we have

$$T_1(D,z)=V_1(D,s)\le V(z)\bigl(T_1(s)+O(Ks^{\widehat{\kappa}-\kappa+1}\widehat{T}^{+}(s)(\log D)^{-1})\bigr). \tag{14.8}$$

This is stronger than the assertion and proves the case $N=1$.

Induction step from $N-1$ to N with $N\ge2$. For the sum $V_n(D,z)$, we have

$$s>\beta+n\implies V_n(D,z)=0$$

and so we may assume $s\le\beta+N$. By Claim 14.5, the assertion follows when (14.5) with C depending on C_1. From now on, in this induction step, every implicit constant is independent of C_1. We subdivide the remaining case into two cases:

- Case I. When $\beta+\varepsilon_N\le s\le\sigma$ and $\log D\ge C_1K^{\Theta_\kappa}$.
- Case II. When $\beta-1<s\le\beta+1$, $\log D\ge C_1K^{\Theta_\kappa}$ and N is odd.

where C_1 is some large constant $C_1=C_1(\kappa,\Delta)$.

Case I. When $\beta+\varepsilon_N\le s\le\sigma$ and $\log D\ge C_1K^{\Theta_\kappa}$. We have

$$z_n:=D^{\min(\frac{1}{s},\frac{1}{\beta+\varepsilon_n})}=D^{\min(\frac{1}{s},\frac{1}{\beta+\varepsilon_N})}=D^{\frac{1}{s}}=z\quad\text{for }n\equiv N\ (\mathrm{mod}\ 2)$$

and

$$V_1(D,z)=0\quad\text{if }N\text{ is odd}.$$

By Lemma 7.1, we have

$$V_n(D,z)=\sum_{y_n\le p<z}\omega(p)V_{n-1}\biggl(\frac{D}{p},p\biggr)\quad\text{for }n\ge2\text{ with }n\equiv N\ (\mathrm{mod}\ 2).$$

Note that

$$p < y_n \implies \frac{\log \frac{D}{p}}{\log p} = \frac{\log D}{\log p} - 1 > \beta + n - 1 \implies V_{n-1}\left(\frac{D}{p}, p\right) = 0$$

and so we can write

$$V_n(D, z) = \sum_{p<z} \omega(p) V_{n-1}\left(\frac{D}{p}, p\right) \quad \text{for } n \geq 2 \text{ with } n \equiv N \pmod 2.$$

By taking the sum over n, we have

$$\begin{aligned} T_N(D, z) &= \sum_{\substack{1 \leq n \leq N \\ n \equiv N \ (\mathrm{mod}\ 2)}} V_n(D, z) \\ &= \sum_{\substack{2 \leq n \leq N \\ n \equiv N \ (\mathrm{mod}\ 2)}} V_n(D, z) \\ &= \sum_{p<z} \omega(p) \sum_{\substack{2 \leq n \leq N \\ n \equiv N \ (\mathrm{mod}\ 2)}} V_{n-1}\left(\frac{D}{p}, p\right) \\ &= \sum_{p<z} \omega(p) \sum_{\substack{1 \leq n \leq N-1 \\ n \equiv N-1 \ (\mathrm{mod}\ 2)}} V_n\left(\frac{D}{p}, p\right) \\ &= \sum_{p<z} \omega(p) T_{N-1}\left(\frac{D}{p}, p\right). \end{aligned}$$

Furthermore, note that the parameter z appears on the right-hand side only in the summation condition on p. Thus, we can decompose as

$$T_N(D, z) = \sum\nolimits_0 + \sum\nolimits_1 + \sum\nolimits_2, \tag{14.9}$$

where

$$\begin{aligned} \sum\nolimits_0 &\coloneqq \sum_{p < D^{\frac{1}{\sigma}}} \omega(p) T_{N-1}\left(\frac{D}{p}, p\right) = T_N(D, D^{\frac{1}{\sigma}}), \\ \sum\nolimits_1 &\coloneqq \sum_{D^{\frac{1}{\sigma}} \leq p < D^{\frac{1}{\tau}}} \omega(p) T_{N-1}\left(\frac{D}{p}, p\right), \\ \sum\nolimits_2 &\coloneqq \sum_{D^{\frac{1}{\tau}} \leq p < z} \omega(p) T_{N-1}\left(\frac{D}{p}, p\right). \end{aligned}$$

and

$$\tau \coloneqq \max\left(s, \left(1 - \frac{\log 2}{\log D}\right)^{-1}\right) > 1 \quad \text{and so} \quad D^{\frac{1}{\tau}} = \min(z, \tfrac{D}{2}).$$

We first consider the sum

$$\sum\nolimits_1 = \sum_{D^{\frac{1}{\sigma}} \leq p < D^{\frac{1}{\tau}}} \omega(p) T_{N-1}\left(\frac{D}{p}, p\right).$$

Note that we may assume $\sigma \geq \tau \geq s$ since otherwise $\sum_1$ is empty. Since

$$p < z \text{ and } s \geq \beta + \varepsilon_N \implies \frac{\log \frac{D}{p}}{\log p} = \frac{\log D}{\log p} - 1 \in I_{N-1},$$

$$D/2 = D^{1-\frac{\log 2}{\log D}}$$

and $p < D/2$ assures $D/p \geq 2$, we can use the induction hypothesis to obtain

$$\sum\nolimits_1 \leq \sum\nolimits_{11} + \sum\nolimits_{12} \tag{14.10}$$

where

$$\sum\nolimits_{11} \coloneqq V(z) \sum_{D^{\frac{1}{\sigma}} \leq p < D^{\frac{1}{\tau}}} \omega(p) \frac{V(p)}{V(z)} T_{N-1}\left(\frac{\log D}{\log p} - 1\right),$$

$$\sum\nolimits_{12} \coloneqq Ce^{\sqrt{K}} V(z) \sum_{D^{\frac{1}{\sigma}} \leq p < D^{\frac{1}{\tau}}} \omega(p) \frac{V(p)}{V(z)} E_{N-1}\left(\frac{D}{p}, \frac{\log D}{\log p} - 1\right)\left(\log \frac{D}{p}\right)^{-\Delta}$$

with

$$\tau \coloneqq \max\left(s, \left(1 - \frac{\log 2}{\log D}\right)^{-1}\right).$$

Note that we have

$$\left(1 - \frac{\log 2}{\log D}\right)^{-1} \leq 2 \ll s \quad \text{and so} \quad \tau \ll s$$

if $D \geq C_1 \geq 4$. We estimate the sums $\sum_{11}$ and $\sum_{12}$ separately.

For the sum $\sum_{11}$, we use Lemma 8.7. By Proposition 9.3, we have

$$t^\kappa T_{N-1}(t-1) = \left(1 + \frac{1}{t-1}\right)^\kappa (t-1)^\kappa T_{N-1}(t-1)$$

is decreasing and continuous for $t \geq \tau > 1$. Thus, Lemma 8.7 gives

$$\sum\nolimits_{11} \leq V(z)\left(\int_s^\sigma T_{N-1}(t-1)\frac{dt^\kappa}{s^\kappa} + \frac{3(\kappa+1)K^2 T_{N-1}(\tau-1)}{\log D^{\frac{1}{\sigma}}}\left(\frac{\tau}{s}\right)^\kappa\right).$$

By (iii) of Proposition 9.3, we have

$$\sum\nolimits_{11} \leq V(z)\left(T_N(s) + \frac{(\kappa+1)K^2 \sigma T_{N-1}(\tau-1)}{\log D}\left(\frac{\tau}{s}\right)^\kappa\right).$$

By Lemma 13.2 and (i), (iii) of Proposition 13.1 gives

$$\begin{aligned} T_{N-1}(\tau-1)\left(\frac{\tau}{s}\right)^\kappa &\ll (\tau-1)^{\widehat{\kappa}-\kappa+1}\widehat{T}^{\mp}(\tau-1)\left(\frac{\tau}{s}\right)^\kappa \\ &\ll (\tau-1)^{-\kappa}(\tau\log e\tau)\tau^{\widehat{\kappa}+1}\widehat{T}^{\mp}(\tau)\left(\frac{\tau}{s}\right)^\kappa \\ &\ll \left(\frac{\tau}{\tau-1}\right)^\kappa (\sigma\log e\sigma)s^{\widehat{\kappa}-\kappa+1}\widehat{T}^{\pm}(s) \end{aligned}$$

since $t^{\widehat{\kappa}+1}\widehat{T}^{\pm}(t)$ is decreasing. Thus, we have

$$\sum\nolimits_{11} \leq V(z)\left(T_N(s) + O\left(\left(\frac{\tau}{\tau-1}\right)^\kappa (\sigma^2\log e\sigma)\frac{K^2 s^{\widehat{\kappa}-\kappa+1}\widehat{T}^{\pm}(s)}{\log D}\right)\right).$$

When $\kappa > \frac{1}{2}$, we have

$$\tau \geq s \geq \beta > 1 \quad \text{and so} \quad \left(\frac{\tau}{\tau - 1}\right)^{\kappa} \ll 1.$$

When $\kappa \leq \frac{1}{2}$, we have

$$\tau \geq \left(1 - \frac{\log 2}{\log D}\right)^{-1} \quad \text{and so} \quad \left(\frac{\tau}{\tau - 1}\right)^{\kappa} \leq \left(\frac{\log D}{\log 2}\right)^{\kappa} \ll (\log D)^{\kappa}.$$

Thus, in any case, we have

$$\left(\frac{\tau}{\tau - 1}\right)^{\kappa} \ll (\log D)^{(1-\Delta_0)}. \tag{14.11}$$

We thus have

$$\sum\nolimits_{11} \leq V(z)\bigg(T_N(\tau) + O\bigg((\sigma^3 \log e\sigma)\frac{K^2}{\sigma}\tau^{\widehat{\kappa}-\kappa+1}\widehat{T}^{\pm}(s)(\log D)^{-\Delta_0}\bigg)\bigg).$$

(The denominator σ in the error term is kept for later necessity.) Note that

$$\sigma^3 \log e\sigma \ll (\log D)^{\frac{3}{d}}(\log\log D)^4.$$

Thus, we arrive at

$$\sum\nolimits_{11} \leq V(z)\bigg(T_N(s) + O\bigg(\frac{1}{\log\log D}\frac{K^2}{\sigma}E_N(D,z)(\log D)^{-\Delta}\bigg)\bigg) \tag{14.12}$$

since

$$\frac{3}{d} < \Delta_0 - \Delta$$

assured by (14.1). This completes the estimate of $\sum_{11}$.

For the sum $\sum_{12}$, by writing

$$s_p := \frac{\log D}{\log p},$$

note that

$$\begin{aligned}
&E_{N-1}\bigg(\frac{D}{p}, \frac{\log D}{\log p} - 1\bigg)\bigg(\log\frac{D}{p}\bigg)^{-\Delta} \\
&= (\log D)^{-\Delta}E_{N-1}\bigg(\frac{D}{p}, s_p - 1\bigg)\bigg(\frac{s_p}{s_p - 1}\bigg)^{\Delta} \\
&= (\log D)^{-\Delta}\bigg(1 + \frac{(s_p - 1)^d}{\log\frac{D}{p}}\bigg)^{s_p - 1}(s_p - 1)^{\widehat{\kappa}-\kappa+1}\widehat{T}^{\mp}(s_p - 1)\bigg(\frac{s_p}{s_p - 1}\bigg)^{\Delta}.
\end{aligned}$$

Since

$$\log\frac{D}{p} = \left(\frac{\log\frac{D}{p}}{\log D}\right)\log D = \left(\frac{s_p - 1}{s_p}\right)\log D$$

when $D^{\frac{1}{s}} \leq p < D^{\frac{1}{\tau}}$, we now have

$$\bigg(1 + \frac{(s_p - 1)^d}{\log\frac{D}{p}}\bigg)^{s_p - 1} \leq \bigg(1 + \frac{s_p(s_p - 1)^{d-1}}{\log D}\bigg)^{s_p - 1} \leq \bigg(1 + \frac{s_p^d}{\log D}\bigg)^{s_p - 1}$$

and so

$$E_{N-1}\left(\frac{D}{p}, \frac{\log D}{\log p} - 1\right)\left(\log \frac{D}{p}\right)^{-\Delta}$$
$$\leq (\log D)^{-\Delta}\left(1 + \frac{s_p^d}{\log D}\right)^{s_p-1}(s_p - 1)^{\widehat{\kappa}-\kappa+1}\widehat{T}^{\mp}(s_p - 1)\left(\frac{s_p}{s_p - 1}\right)^{\Delta}.$$

Therefore, by writing

$$q_D^{\pm}(s) := \left(1 + \frac{s^d}{\log D}\right)^{s-1}(s-1)^{\widehat{\kappa}-\kappa+1}\widehat{T}^{\pm}(s-1)\left(\frac{s}{s-1}\right)^{\Delta},$$

we have

$$E_{N-1}\left(\frac{D}{p}, \frac{\log D}{\log p} - 1\right)\left(\log \frac{D}{p}\right)^{-\Delta} \leq (\log D)^{-\Delta} q_D^{\mp}\left(\frac{\log D}{\log p}\right).$$

This gives

$$\sum\nolimits_{12} \leq Ce^{\sqrt{K}}V(z)(\log D)^{-\Delta} \sum_{D^{\frac{1}{\sigma}} \leq p < D^{\frac{1}{\tau}}} \omega(p)\frac{V(p)}{V(z)} q_D^{\mp}\left(\frac{\log D}{\log p}\right). \tag{14.13}$$

We use Lemma 8.6. We need to check $q_D^{\mp}(t)t^{\kappa}$ is decreasing for $\tau \leq t \leq \sigma$. Since

$$q_D^{\mp}(t)t^{\kappa} = \left(1 + \frac{t^d}{\log D}\right)^{t-1}(t-1)^{\widehat{\kappa}+1}\widehat{T}^{\mp}(t-1)\left(\frac{t}{t-1}\right)^{\kappa+\Delta}, \tag{14.14}$$

it suffices to show

$$\left(1 + \frac{t^d}{\log D}\right)^{t-1}(t-1)^{\widehat{\kappa}+1}\widehat{T}^{\mp}(t-1)$$

is decreasing. For the later necessity, we consider a bit more general function

$$\Lambda_{\varepsilon}^{\pm}(t) := \left(1 + \frac{(t+\varepsilon)^d}{\log D}\right)^{t} t^{\widehat{\kappa}+1}\widehat{T}^{\pm}(t) \quad \text{for } t > 0 \text{ and } \varepsilon = 0, 1.$$

Note that

$$\Lambda_1^{\mp}(t-1) = \left(1 + \frac{t^d}{\log D}\right)^{t-1}(t-1)^{\widehat{\kappa}+1}\widehat{T}^{\mp}(t-1)$$

and so, by recalling (14.14), we have

$$\begin{aligned}&\Lambda_1^{\mp}(t-1)\left(\frac{t}{t-1}\right)^{\kappa+\Delta}\\ &= \left(1 + \frac{t^d}{\log D}\right)^{t-1}(t-1)^{\widehat{\kappa}+1}\widehat{T}^{\mp}(t-1)\left(\frac{t}{t-1}\right)^{\kappa+\Delta} = q_D^{\mp}(t)t^{\kappa}.\end{aligned} \tag{14.15}$$

Claim 14.6. *Assume that D is sufficiently large in terms of κ and Δ.*

(i) *For $\varepsilon = 0, 1$, the function $\Lambda_{\varepsilon}^{\pm}(t)$ is decreasing for $t \in [\widehat{\beta} + \varepsilon_{\pm}, \sigma]$.*
(ii) *The function $q_D^{\mp}(t)t^{\kappa}$ is decreasing for $t \in (\beta + \varepsilon_{\pm}, \sigma]$.*
(iii) *For $s \in [\beta + \varepsilon_{\pm}, \sigma]$, we have*

$$\int_s^{\sigma} q_D^{\mp}(t)dt^{\kappa} < \left(1 - \frac{1}{\sigma}\right)^{1-\Delta}\Lambda_0^{\pm}(s).$$

Proof. We first prepare an estimate of the derivative $(\Lambda_\varepsilon^\pm(t))'$. For $t>0$, we have

$$\begin{aligned}\left(\left(1+\frac{(t+\varepsilon)^d}{\log D}\right)^t\right)' &= \left(1+\frac{(t+\varepsilon)^d}{\log D}\right)^t\left(t\log\left(1+\frac{(t+\varepsilon)^d}{\log D}\right)\right)' \\ &= \left(1+\frac{(t+\varepsilon)^d}{\log D}\right)^t\left(\log\left(1+\frac{(t+\varepsilon)^d}{\log D}\right)+\frac{dt(t+\varepsilon)^{d-1}}{1+\frac{(t+\varepsilon)^d}{\log D}}\frac{1}{\log D}\right) \\ &\le \left(1+\frac{(t+\varepsilon)^d}{\log D}\right)^t\left(\log\left(1+\frac{(t+\varepsilon)^d}{\log D}\right)+d\frac{\frac{(t+\varepsilon)^d}{\log D}}{1+\frac{(t+\varepsilon)^d}{\log D}}\right).\end{aligned}$$

Since

$$\log(1+x)=\int_1^{1+x}\frac{du}{u}\ge\frac{1}{1+x}\int_1^{1+x}dt=\frac{x}{1+x}\quad\text{for } x\ge 0,$$

we further have

$$\left(\left(1+\frac{(t+\varepsilon)^d}{\log D}\right)^t\right)'\le(d+1)\left(1+\frac{(t+\varepsilon)^d}{\log D}\right)^t\log\left(1+\frac{(t+\varepsilon)^d}{\log D}\right) \tag{14.16}$$

for $t>0$. We then consider the range $t>\widehat{\beta}+\varepsilon_\pm$. In this range,

$$(t^{\widehat{\kappa}+1}\widehat{T}^\pm(t))'=-\widehat{\kappa}t^{\widehat{\kappa}}\widehat{T}^\mp(t-1).$$

Thus, by differentiating and using (14.16), we have

$$\begin{aligned}&(\Lambda_\varepsilon^\pm(t))' \\ &=\left(\left(1+\frac{(t+\varepsilon)^d}{\log D}\right)^t t^{\widehat{\kappa}+1}\widehat{T}^\pm(t)\right)' \\ &=\left(1+\frac{(t+\varepsilon)^d}{\log D}\right)^t(t^{\widehat{\kappa}+1}\widehat{T}^\pm(t))'+\left(\left(1+\frac{(t+\varepsilon)^d}{\log D}\right)^t\right)'t^{\widehat{\kappa}+1}\widehat{T}^\pm(t) \\ &\le-\widehat{\kappa}\left(1+\frac{(t+\varepsilon)^d}{\log D}\right)^t t^{\widehat{\kappa}}\widehat{T}^\mp(t-1) \\ &\qquad+(d+1)\left(1+\frac{(t+\varepsilon)^d}{\log D}\right)^t t^{\widehat{\kappa}+1}\widehat{T}^\pm(t)\log\left(1+\frac{(t+\varepsilon)^d}{\log D}\right) \\ &=-\widehat{\kappa}\left(1+\frac{(t+\varepsilon)^d}{\log D}\right)^t t^{\widehat{\kappa}}\widehat{T}^\mp(t-1)\left(1-\frac{d+1}{\widehat{\kappa}}\frac{t\widehat{T}^\pm(t)}{\widehat{T}^\mp(t-1)}\log\left(1+\frac{(t+\varepsilon)^d}{\log D}\right)\right).\end{aligned}$$

By using (i), (iii) of Proposition 13.1 in the form

$$\frac{t\widehat{T}^\pm(t)}{\widehat{T}^\mp(t-1)}\ll\left(\frac{t-1}{t}\right)^{\widehat{\kappa}+1}\frac{1}{\log et}\ll\frac{1}{\log et},$$

we have

$$(\Lambda_\varepsilon^\pm(t))'\le-\widehat{\kappa}\left(1+\frac{(t+\varepsilon)^d}{\log D}\right)^t t^{\widehat{\kappa}}\widehat{T}^\mp(t-1)\big(1+R_\varepsilon^\pm(t)\big)\quad\text{for } t>\widehat{\beta}+\varepsilon_\pm \tag{14.17}$$

with

$$R_\varepsilon^\pm(t)\ll\frac{1}{\log et}\log\left(1+\frac{(t+\varepsilon)^d}{\log D}\right).$$

We now prove the claim.

(i). When $t \le (\log D)^{\frac{1}{2d}}$, we have

$$R_\varepsilon^{\pm}(t) \ll \frac{(t+\varepsilon)^d}{\log D} \ll \log D^{-\frac{1}{2}} \ll (\log\log D)^{-\frac{1}{2}}.$$

When $(\log D)^{\frac{1}{2d}} < t \le \sigma$, we have

$$R_\varepsilon^{\pm}(t) \ll \frac{\log(1+(\log\log D)^d)}{\log\log D} \ll (\log\log D)^{-\frac{1}{2}}.$$

Thus, in any case, we have

$$(\Lambda_\varepsilon^{\pm}(t))' \le -\widehat{\kappa}\left(1+\frac{(t+\varepsilon)^d}{\log D}\right)^t t^{\widehat{\kappa}}\widehat{T}^{\mp}(t-1)\big(1+O((\log\log D)^{-\frac{1}{2}})\big).$$

Thus, for D is sufficiently large in terms of κ and Δ, we have

$$(\Lambda_\varepsilon^{\pm}(t))' < 0.$$

This shows that $\Lambda_\varepsilon^{\pm}(t)$ is decreasing for $t \ge \widehat{\beta}+\varepsilon_{\pm}$.

(ii). By (14.15) and the above proven (ii), $q_D^{\mp}(t)t^{\kappa}$ is decreasing for $t \in [\widehat{\beta}+\varepsilon_{\mp}+1, \sigma]$. Thus, it thus suffices to consider the range $t \in (\beta+\varepsilon_{\pm}, \widehat{\beta}+\varepsilon_{\mp}+1]$. In this range, we have $t-1 \in (\beta+\varepsilon_{\mp}, \widehat{\beta}+\varepsilon_{\mp}]$ and so $(t-1)^{\widehat{\kappa}+1}\widehat{T}^{\mp}(t-1)$ is constant. Therefore, it suffices to show

$$\left(1+\frac{t^d}{\log D}\right)^t\left(\frac{t}{t-1}\right)^{\kappa+\Delta}$$

is decreasing for $t \in (\beta+\varepsilon_{\pm}, \widehat{\beta}+\varepsilon_{\mp}+1]$. By taking the derivative and using (14.16),

$$\begin{aligned}
&\left(\left(1+\frac{t^d}{\log D}\right)^t\left(\frac{t}{t-1}\right)^{\kappa+\Delta}\right)' \\
&= \left(1+\frac{t^d}{\log D}\right)^t\left(\left(\frac{t}{t-1}\right)^{\kappa+\Delta}\right)' + \left(\left(1+\frac{t^d}{\log D}\right)^t\right)'\left(\frac{t}{t-1}\right)^{\kappa+\Delta} \\
&= -\frac{\kappa+\Delta}{(t-1)^2}\left(1+\frac{t^d}{\log D}\right)^t\left(\frac{t}{t-1}\right)^{\kappa+\Delta-1} \\
&\qquad + (d+1)\left(1+\frac{t^d}{\log D}\right)^t\left(\frac{t}{t-1}\right)^{\kappa+\Delta}\log\left(1+\frac{t^d}{\log D}\right) \\
&= -\frac{\kappa+\Delta}{t^2}\left(1+\frac{t^d}{\log D}\right)^t\left(\frac{t}{t-1}\right)^{\kappa+\Delta+1} \\
&\qquad + (d+1)\left(1+\frac{t^d}{\log D}\right)^t\left(\frac{t}{t-1}\right)^{\kappa+\Delta}\log\left(1+\frac{t^d}{\log D}\right) \\
&= -\frac{\kappa+\Delta}{t^2}\left(1+\frac{t^d}{\log D}\right)^t\left(\frac{t}{t-1}\right)^{\kappa+\Delta+1}\left(1+O\left(\frac{t^{d+1}(t-1)}{\log D}\right)\right) < 0
\end{aligned}$$

If D is sufficiently large in terms of κ, Δ. This shows (ii).

(iii). When t is sufficiently large in terms of κ, say $t \ge t_0(\kappa, \Delta)$, we can show

$$1+R_0^{\pm}(t) \ge \left(1+\frac{t^d}{\log D}\right)^{-1}.$$

Indeed, when $t_0 \le t \le (\log D)^{\frac{1}{d}}$, we have

$$1 + R_\varepsilon^\pm(t) \ge 1 - \frac{1}{2}\log\left(1 + \frac{t^d}{\log D}\right) \ge 1 - \frac{1}{2}\frac{t^d}{\log D} \ge \left(1 + \frac{t^d}{\log D}\right)^{-1}.$$

When $(\log D)^{\frac{1}{d}} < t \le \sigma$, we instead have

$$1 + R_\varepsilon^\pm(t) \ge 1 + O\left(\frac{\log(1 + (\log\log D)^d)}{\log\log D}\right) \ge \frac{1}{2} \ge \left(1 + \frac{t^d}{\log D}\right)^{-1}$$

provided D is sufficiently large. Therefore, by (14.17), we have

$$(\Lambda_0^\pm(t))' \le -\widehat{\kappa}\left(1 + \frac{t^d}{\log D}\right)^{t-1} t^{\widehat{\kappa}}\widehat{T}^\mp(t-1) = -\widehat{\kappa}\Lambda_1^\mp(t-1)\left(\frac{t}{t-1}\right)^{\widehat{\kappa}+1}\frac{1}{t}.$$

By (14.15), this gives

$$\begin{aligned}
\kappa q_D^\mp(t)t^{\kappa-1} &= \kappa\Lambda_1^\mp(t-1)\left(\frac{t}{t-1}\right)^{\kappa+\Delta}\frac{1}{t} \\
&\le -\frac{\kappa}{\widehat{\kappa}}(\Lambda_0^\pm(t))'\left(\frac{t-1}{t}\right)^{\widehat{\kappa}-\kappa+1-\Delta} \\
&\le -(\Lambda_0^\pm(t))'\left(\frac{t-1}{t}\right)^{1-\Delta} \qquad \text{for } t \ge t_0.
\end{aligned}$$

Therefore, we have (iii) in the range $s \ge t_0$ as

$$\begin{aligned}
\int_s^\sigma q_D^\mp(t)dt^\kappa &\le -\int_s^\sigma (\Lambda_0^\pm(t))'\left(\frac{t-1}{t}\right)^{1-\Delta}dt \\
&\le -\left(1 - \frac{1}{\sigma}\right)^{1-\Delta}\int_s^\sigma (\Lambda_0^\pm(t))'dt \\
&< \left(1 - \frac{1}{\sigma}\right)^{1-\Delta}\Lambda_0^\pm(s).
\end{aligned}$$

We next consider the range $\beta + \varepsilon_\pm \le s \le t_0$. By the above proven case, we have

$$\begin{aligned}
\int_{t_0+2}^\sigma q_D^\mp(t)dt^\kappa &< \left(1 - \frac{1}{\sigma}\right)^{1-\Delta}\Lambda_0^\pm(t_0+2) \\
&= \left(1 - \frac{1}{\sigma}\right)^{1-\Delta}\left(1 + \frac{(t_0+2)^d}{\log D}\right)^{t_0+2}(t_0+2)^{\widehat{\kappa}+1}\widehat{T}^\pm(t_0+2).
\end{aligned}$$

By (iii) of Proposition 13.1, we then obtain

$$\begin{aligned}
\int_{t_0+2}^\sigma q_D^\mp(t)dt^\kappa &\ll \left(1 - \frac{1}{\sigma}\right)^{1-\Delta}\frac{1}{(t_0\log et_0)^2}\left(1 + \frac{(t_0+2)^d}{\log D}\right)^{t_0+2}t_0^{\widehat{\kappa}+1}\widehat{T}^\pm(t_0) \\
&\ll \left(1 - \frac{1}{\sigma}\right)^{1-\Delta}\frac{1}{(t_0\log et_0)^2}\left(1 + \frac{(t_0+2)^d}{\log D}\right)^{t_0+2}s^{\widehat{\kappa}+1}\widehat{T}^\pm(s)
\end{aligned}$$

When D is sufficiently large in terms of κ, Δ, t_0, we thus have

$$\int_{t_0+2}^\sigma q_D^\mp(t)dt^\kappa \ll \frac{1}{t_0^2}\left(1 - \frac{1}{\sigma}\right)^{1-\Delta}s^{\widehat{\kappa}+1}\widehat{T}^\pm(s).$$

By (14.2) as we made available by choosing δ small, we have

$$\begin{aligned}\int_s^{t_0+2} q_D^{\mp}(t)dt^\kappa &= \kappa\int_s^{t_0+2}\left(1+\frac{t^d}{\log D}\right)^{t-1}(t-1)^{\widehat{\kappa}+1}\widehat{T}^{\mp}(t-1)\left(\frac{t}{t-1}\right)^{\kappa+\Delta}\frac{dt}{t}\\ &\le \left(1+\frac{(t_0+2)^d}{\log D}\right)^{t_0+1}\kappa\int_s^{t_0+2}\left(\frac{t-1}{t}\right)^{1-\Delta}t^{\widehat{\kappa}}\widehat{T}^{\mp}(t-1)dt\\ &\le \left(1-\frac{1}{t_0+2}\right)^{\frac12(\Delta_0-\Delta)}\left(1+\frac{(t_0+2)^d}{\log D}\right)^{t_0+1}\\ &\qquad\times\kappa\int_s^{t_0+2}\left(\frac{t-1}{t}\right)^{1-\Delta_0+\frac12(\Delta_0-\Delta)}t^{\widehat{\kappa}}\widehat{T}^{\mp}(t-1)dt\\ &\le \left(1-\frac{1}{t_0+2}\right)^{\frac12(\Delta_0-\Delta)}\left(1+\frac{(t_0+2)^d}{\log D}\right)^{t_0+2}s^{\widehat{\kappa}+1}\widehat{T}^{\pm}(s).\end{aligned}$$

When D is sufficiently large in terms of κ, Δ, t_0, we thus have

$$\int_s^{t_0+2} q_D^{\mp}(t)dt^\kappa \le \left(1-\frac{1}{\sigma}\right)^{1-\Delta}\left(1-\frac{1}{t_0+2}\right)^{\frac13(\Delta-\Delta_0)}\left(1+O\left(\frac{1}{t_0^2}\right)\right)s^{\widehat{\kappa}+1}\widehat{T}^{\pm}(s)$$

since

$$\left(1-\frac{1}{t_0+2}\right)^{\frac16(\Delta-\Delta_0)}\ge 1-\frac{1}{t_0+2}\ge 1-\frac{1-\Delta}{\sigma}\ge\left(1-\frac{1}{\sigma}\right)^{1-\Delta}.$$

By combining the above estimates, we obtain

$$\begin{aligned}\int_s^{\sigma} q_D^{\mp}(t)dt^\kappa &\le \left(1-\frac{1}{\sigma}\right)^{1-\Delta}s^{\widehat{\kappa}+1}\widehat{T}^{\pm}(s)\left\{\left(1-\frac{1}{t_0+2}\right)^{\frac13(\Delta-\Delta_0)}+O\left(\frac{1}{t_0^2}\right)\right\}\\ &\le \left(1-\frac{1}{\sigma}\right)^{1-\Delta}s^{\widehat{\kappa}+1}\widehat{T}^{\pm}(s)\left\{1-\frac13\frac{\Delta-\Delta_0}{t_0+2}+O\left(\frac{1}{t_0^2}\right)\right\}\\ &\le \left(1-\frac{1}{\sigma}\right)^{1-\Delta}s^{\widehat{\kappa}+1}\widehat{T}^{\pm}(s)\le\left(1-\frac{1}{\sigma}\right)^{1-\Delta}\Lambda_0^{\pm}(s)\end{aligned}$$

by replacing t_0 large enough in terms of κ, Δ. This proves (iii) for all $s \ge \beta+\varepsilon_N$. □

By (ii) of Claim 14.6, we can apply Lemma 8.7 to (14.13). This gives

$$\sum\nolimits_{12}\le Ce^{\sqrt{K}}V(z)(\log D)^{-\Delta}\left(\int_s^{\sigma} q_D^{\mp}(t)\frac{dt^\kappa}{s^\kappa}+\frac{3(\kappa+1)K^2q_D^{\mp}(\tau)}{\log D^{\frac1\sigma}}\left(\frac{\tau}{s}\right)^{\kappa}\right).$$

By (iii) of Claim 14.6, this gives

(14.18)

$$\begin{aligned}&\sum\nolimits_{12}\\ &\le Ce^{\sqrt{K}}V(z)(\log D)^{-\Delta}\\ &\quad\times\left(\left(1-\frac{1}{\sigma}\right)^{1-\Delta}s^{-\kappa}\Lambda_0^{\pm}(s)+\frac{3(\kappa+1)K^2q_D^{\mp}(\tau)}{\log D^{\frac1\sigma}}\left(\frac{\tau}{s}\right)^{\kappa}\right)\\ &\le Ce^{\sqrt{K}}V(z)(\log D)^{-\Delta}\\ &\quad\times\left(\left(1-\frac{1}{\sigma}\right)^{1-\Delta}\left(1+\frac{s^d}{\log D}\right)^{s}s^{\widehat{\kappa}-\kappa+1}\widehat{T}^{\pm}(s)+\frac{3(\kappa+1)K^2q_D^{\mp}(\tau)}{\log D^{\frac1\sigma}}\left(\frac{\tau}{s}\right)^{\kappa}\right).\end{aligned}$$

By (14.15) and (iii) of Proposition 13.1, we have

$$\begin{aligned}\frac{q_D^{\mp}(\tau)}{\log D^{\frac{1}{\sigma}}}\left(\frac{\tau}{s}\right)^{\kappa} &= \frac{\sigma}{\log D}\left(1+\frac{\tau^d}{\log D}\right)^{\tau-1}(\tau-1)^{\widehat{\kappa}+1}\widehat{T}^{\pm}(\tau-1)\left(\frac{\tau}{\tau-1}\right)^{\kappa+\Delta}s^{-\kappa}\\ &\ll \frac{\sigma^2\log e\sigma}{(\log D)}\left(1+\frac{\tau^d}{\log D}\right)^{\tau}\tau^{\widehat{\kappa}+1}\widehat{T}^{\pm}(\tau)\left(\frac{\tau}{\tau-1}\right)^{\kappa+\Delta}s^{-\kappa}\\ &\ll \frac{\sigma^2\log e\sigma}{(\log D)}\left(1+\frac{s^d}{\log D}\right)^{s}s^{\widehat{\kappa}-\kappa+1}\widehat{T}^{\pm}(s)\left(\frac{\tau}{\tau-1}\right)^{\kappa+\Delta}.\end{aligned}$$

by the monotonicity of $t^{\widehat{\kappa}+1}\widehat{T}^{\pm}(t)$ and $\tau = s$ when $s \geq 2$. By (14.11), we have

$$\begin{aligned}\left(\frac{\tau}{\tau-1}\right)^{\kappa+\Delta} &= \left(\frac{\tau}{\tau-1}\right)^{\kappa}\left(\frac{\tau}{\tau-1}\right)^{\Delta}\\ &\leq \left(\frac{\tau}{\tau-1}\right)^{\kappa}\left(\frac{\log D}{\log 2}\right)^{\Delta} \ll (\log D)^{1-(\Delta_0-\Delta)}\end{aligned}$$

and so

$$\begin{aligned}K^2\frac{q_D^{\mp}(\tau)}{\log D^{\frac{1}{\sigma}}}\left(\frac{\tau}{s}\right)^{\kappa} &\ll \frac{K^2\sigma^3\log e\sigma}{\sigma(\log D)^{\Delta_0-\Delta}}\left(1+\frac{s^d}{\log D}\right)^{s}s^{\widehat{\kappa}-\kappa+1}\widehat{T}^{\pm}(s)\\ &\ll \frac{1}{\log\log D}\frac{1}{\sigma}\left(1+\frac{s^d}{\log D}\right)^{s}s^{\widehat{\kappa}-\kappa+1}\widehat{T}^{\pm}(s)\end{aligned}$$

since

$$\frac{2}{\Theta_K}+\frac{3}{d} < \Delta_0-\Delta.$$

as in (14.4). On inserting this estimate into (14.18), we obtain

$$\begin{aligned}\textstyle\sum_{12} &\leq Ce^{\sqrt{K}}V(z)E_N(D,z)(\log D)^{-\Delta}\\ &\qquad\times\left(\left(1-\frac{1}{\sigma}\right)^{1-\Delta}+O\left(\frac{1}{\log\log D}\frac{1}{\sigma}\right)\right).\end{aligned}\tag{14.19}$$

This completes the estimate of $\sum_{12}$.

We next consider the sum

$$\textstyle\sum_0 = T_N(D, D^{\frac{1}{\sigma}}).$$

For this sum, we can use Claim 14.5 with $z = D^{\frac{1}{\sigma}}$ to obtain

$$\textstyle\sum_0 \ll V(D)\cdot\frac{1}{\log D}\frac{e^{\sqrt{K}}}{\sigma}E_N(D,\sigma)(\log D)^{-\Delta}.$$

When $s \geq \widehat{\beta}+\varepsilon_{\pm}$, by (ii) of Claim 14.6, we have

$$E_N(D,\sigma) = \sigma^{-\kappa}\Lambda_0^{\pm}(\sigma) \leq \sigma^{-\kappa}\Lambda_0^{\pm}(s) \leq E_N(D,s).$$

When $\beta+\varepsilon_N \leq s \leq \widehat{\beta}+\varepsilon_{\pm}$, we also have

$$E_N(D,\sigma) = \sigma^{-\kappa}\Lambda_0^{\pm}(\sigma) \leq \sigma^{-\kappa}\Lambda_0^{\pm}(\widehat{\beta}+\varepsilon_{\pm}) \ll \sigma^{-\kappa}\Lambda_0^{\pm}(s) \leq E_N(D,s).$$

We therefore have

$$\textstyle\sum_0 \ll V(z)\cdot\frac{1}{\log D}\frac{e^{\sqrt{K}}}{\sigma}E_N(D,s)(\log D)^{-\Delta}\tag{14.20}$$

since $\sigma \geq s$ in the current case. This completes the estimate of $\sum_0$.

We then consider the sum

$$\sum\nolimits_2 = \sum_{D^{\frac{1}{\tau}} \leq p < z} \omega(p) T_{N-1}\left(\frac{D}{p}, p\right).$$

This sum is indeed empty unless $D^{\frac{1}{\tau}} = \min(z, D/2) = D/2$. We thus may assume

$$D^{\frac{1}{s}} = z > D/2 = D^{1-\frac{\log 2}{\log D}}$$

and so

$$\beta + \varepsilon_N \leq s < \frac{1}{1 - \frac{\log 2}{\log D}} \leq 2 \tag{14.21}$$

assuming $D \geq 4$. This happens only if N is even. Also, since

$$\beta > 1 \quad \text{if } \kappa > \frac{1}{2},$$

this situation happens only if $\kappa \leq \frac{1}{2}$ for large $D \geq D(\kappa)$. Thus, we may assume $\min(z, D/2) = D/2$, the bound (14.21) holds, N is even and $\kappa \leq \frac{1}{2}$. Also, if $D/2 \leq p$, we have $D/p \leq 2$. Therefore, the summation condition $p_1 \cdot p_1^{\beta} < D$ makes

$$V_n\left(\frac{D}{p}, p\right) = 0 \quad \text{for odd } n \text{ with } 3 \leq n \leq N-1.$$

Therefore, we have

$$\sum\nolimits_2 = \sum_{D/2 \leq p < z} \omega(p) V_1\left(\frac{D}{p}, p\right).$$

For $D/2 \leq p$, we further have

$$0 \leq V_1\left(\frac{D}{p}, p\right) = \sum_{(D/p)^{\frac{1}{\beta+1}} \leq q < p} \omega(q) V(q) = \sum_{q<p} \omega(q) V(q) = 1 - V(p) \leq 1.$$

Therefore, we have

$$\sum\nolimits_2 \leq \sum_{D/2 \leq p < z} \omega(p) \leq \sum_{D/2 \leq p < z} \log(1 - \omega(p))^{-1} = \log \frac{V(\frac{D}{2})}{V(z)}.$$

Since $\omega \in \Omega(\kappa, K)$, by using (14.21) and $s \geq \beta + \varepsilon_n \geq 1$, we have

$$\begin{aligned}
\sum\nolimits_2 &\leq \kappa \log \frac{\log z}{\log \max(2, \frac{D}{2})} + \log\left(1 + \frac{K}{\log \max(2, \frac{D}{2})}\right) \\
&\leq \kappa \log \frac{\log z}{\log \frac{D}{2}} + \log\left(1 + \frac{K}{\log \frac{D}{2}}\right) \\
&\leq \kappa \log\left(1 - \frac{\log 2}{\log D}\right)^{-1} + \frac{K}{\log D} \ll \frac{K}{\log D}
\end{aligned}$$

provided $D \geq 4$. Since $\kappa \leq \frac{1}{2}$ and $s \geq 1$, we have

$$V(z)^{-1} \ll K(\log D)^{\kappa} \leq K(\log D)^{\frac{1}{2}} = K(\log D)^{1-\Delta_0}$$

By recalling $s \leq 2$ by (14.21), we arrive at

$$\sum\nolimits_2 \ll K(\log D)^{-1} = V(z) K V(z)^{-1} (\log D)^{-1} \ll K^2 \widehat{T}^{-}(s) (\log D)^{-\Delta_0}$$

and so

$$
\begin{aligned}
\textstyle\sum_2 &\ll V(z)\cdot K^2\widehat{T}^{\pm}(s)(\log D)^{-\Delta_0}\\
&= V(z)\cdot K^2\widehat{T}^{\pm}(s)(\log D)^{-\Delta}(\log D)^{-(\Delta_0-\Delta)}\\
&\ll V(z)\cdot \frac{1}{\log\log D}\frac{e^{\sqrt{K}}}{\sigma}E_N(D,z)(\log D)^{-\Delta}.
\end{aligned}
\tag{14.22}
$$

This completes the estimate of $\sum_2$.

By (14.9), (14.10), (14.12), (14.19), (14.20) and (14.22), we have

$$
\begin{aligned}
&T_N(D,z)\\
&\le V(z)\Big\{T_N(s)+\frac{Ce^{\sqrt{K}}E_N(D,z)}{(\log D)^{\Delta}}\Big(\Big(1-\frac{1}{\sigma}\Big)^{1-\Delta}+O\Big(\frac{1}{\log\log D}\frac{1}{\sigma}\Big)\Big)\Big\}.
\end{aligned}
\tag{14.23}
$$

Since

$$
\Big(1-\frac{1}{\sigma}\Big)^{1-\Delta}+O\Big(\frac{1}{\log\log D}\frac{1}{\sigma}\Big)\le 1-\frac{1-\Delta}{\sigma}+O\Big(\frac{1}{\log\log D}\frac{1}{\sigma}\Big)<1
$$

for sufficiently large C_1, we obtain the assertion for the N-th case.

Case II. When $\beta-1<s\le\beta+1$, $\log D\ge C_1K^{\Theta_K}$ and N is odd. In this case,

$$
T_N(D,z)=T_N(D,D^{\frac{1}{\beta+1}})+V_1(D,z)
\tag{14.24}
$$

We apply the above proven (14.23) to $T_N(D,D^{\frac{1}{\beta+1}})$ to obtain

$$
\begin{aligned}
&T_N(D,D^{\frac{1}{\beta+1}})\\
&\le V(D^{\frac{1}{\beta+1}})\\
&\quad\times\Big\{T_N(\beta+1)+\frac{Ce^{\sqrt{K}}E_N(D,D^{\frac{1}{\beta+1}})}{(\log D)^{\Delta}}\Big(\Big(1-\frac{1}{\sigma}\Big)^{1-\Delta}+O\Big(\frac{1}{\log\log D}\frac{1}{\sigma}\Big)\Big)\Big\}.
\end{aligned}
$$

Since $\omega\in\Omega(\kappa,K)$, by using Lemma 13.2, we have

$$
\begin{aligned}
&V(D^{\frac{1}{\beta+1}})T_N(\beta+1)\\
&\le V(z)\Big(\Big(\frac{\beta+1}{s}\Big)^{\kappa}T_N(\beta+1)+\frac{K}{\log D}\frac{(\beta+1)^{\kappa+1}}{s^{\kappa}}T_N(\beta+1)\Big)\\
&\le V(z)\Big(\Big(\frac{\beta+1}{s}\Big)^{\kappa}T_N(\beta+1)+\frac{K(\beta+1)}{\log D}T_N(s)\Big)\\
&= V(z)\Big(\Big(\frac{\beta+1}{s}\Big)^{\kappa}T_N(\beta+1)+O\Big(\frac{1}{\log\log D}\frac{e^{\sqrt{K}}}{\sigma}E_N(D,s)(\log D)^{-\Delta}\Big)\Big)
\end{aligned}
\tag{14.25}
$$

provided

$$
\frac{1}{d}<1-\Delta
$$

assured by (14.1). We also have

$$
\begin{aligned}
E_N(D,\beta+1)&=\Big(1+\frac{\beta+1}{\log D}\Big)^{\beta+1}(\beta+1)^{\widehat{\kappa}-\kappa+1}\widehat{T}^{\pm}(\beta+1)\\
&=\exp\Big((\beta+1)\log\Big(1+\frac{\beta+1}{\log D}\Big)\Big)(\beta+1)^{\widehat{\kappa}-\kappa+1}\widehat{T}^{\pm}(\beta+1)
\end{aligned}
$$

$$
\begin{aligned}
&= \exp\left(s\log\left(1+\frac{s}{\log D}\right)+O\left(\frac{1}{\log D}\right)\right)s^{\widehat{\kappa}-\kappa+1}\widehat{T}^{\pm}(s)\left(\frac{s}{\beta+1}\right)^{\kappa}\\
&\le \left(\frac{s}{\beta+1}\right)^{\kappa}E_N(D,s)\left(1+O\left(\frac{1}{\log D}\right)\right)\\
&= \left(\frac{s}{\beta+1}\right)^{\kappa}E_N(D,s)\left(1+O\left(\frac{1}{\log\log D}\frac{1}{\sigma}\right)\right)
\end{aligned}
$$

by the monotonicity of $s^{\widehat{\kappa}+1}\widehat{T}^{+}(s)$ and have

$$
\begin{aligned}
V(D^{\frac{1}{\beta+1}}) &\le V(z)\left(\frac{\beta+1}{s}\right)^{\kappa}\left(1+\frac{Ks}{\log D}\right)\\
&= V(z)\left(\frac{\beta+1}{s}\right)^{\kappa}\left(1+O\left(\frac{1}{\log\log D}\frac{1}{\sigma}\right)\right)
\end{aligned}
$$

since $\omega \in \Omega(\kappa, K)$ provided

$$\frac{1}{\Theta_K}+\frac{1}{d}<1.$$

Therefore,

$$
\begin{aligned}
&V(D^{\frac{1}{\beta+1}})\frac{Ce^{\sqrt{K}}E_N(D,\beta+1)}{(\log D)^{\Delta}}\left(\left(1-\frac{1}{\sigma}\right)^{1-\Delta}+O\left(\frac{1}{\log\log D}\frac{1}{\sigma}\right)\right)\\
&\le V(z)\frac{Ce^{\sqrt{K}}E_N(D,s)}{(\log D)^{\Delta}}\left(\left(1-\frac{1}{\sigma}\right)^{1-\Delta}+O\left(\frac{1}{\log\log D}\frac{1}{\sigma}\right)\right).
\end{aligned}\tag{14.26}
$$

By (14.25) and (14.26), we have

$$
\begin{aligned}
&T_N(D, D^{\frac{1}{\beta+1}})\\
&\le V(z)\left(\left(\frac{\beta+1}{s}\right)^{\kappa}T_N(\beta+1)\right.\\
&\qquad\left.+\frac{Ce^{\sqrt{K}}E_N(D,s)}{(\log D)^{\Delta}}\left(\left(1-\frac{1}{\sigma}\right)^{1-\Delta}+O\left(\frac{1}{\log\log D}\frac{1}{\sigma}\right)\right)\right)
\end{aligned}\tag{14.27}
$$

For $V_1(D, s)$, we use (14.8) to obtain

$$V_1(D,s)\le V(z)\left(T_1(s)+O\left(\frac{1}{\log\log D}\frac{1}{\sigma}E_N(D,s)(\log D)^{-\Delta}\right)\right)\tag{14.28}$$

provided

$$\frac{1}{\Theta_K}+\frac{1}{d}<1-\Delta.$$

By combining (14.24), (14.27) and (14.28) and using

$$\left(\frac{\beta+1}{s}\right)^{\kappa}T_N(\beta+1)+T_1(s)=T_N(s),$$

which holds since N is odd, we have

$$T_N(D,z)\le V(z)\left\{T_N(s)+\frac{Ce^{\sqrt{K}}E_N(D,z)}{(\log D)^{\Delta}}\left(\left(1-\frac{1}{\sigma}\right)^{1-\Delta}+O\left(\frac{1}{\log\log D}\frac{1}{\sigma}\right)\right)\right\}.$$

Thus, for sufficiently large C_1, we obtain the assertion. □

Lemma 14.7. *For $D \geq z \geq 2$, $\Delta \in (0, \Delta_0)$ with*

$$\Delta_0 = \Delta_0(\kappa) = \begin{cases} 1 & \text{if } \kappa > \frac{1}{2}, \\ \frac{1}{2} & \text{if } 0 < \kappa \leq \frac{1}{2} \end{cases}$$

and a real number d with

$$d > \frac{7}{\Delta_0 - \Delta}$$

and $\omega \in \Omega(\kappa, K)$, we have

$$\begin{cases} V^+(D, z) \leq V(z)\big(F^+(s) + Ce^{\sqrt{K}}E(s)(\log D)^{-\Delta}\big), \\ V^-(D, z) \geq V(z)\big(F^-(s) - Ce^{\sqrt{K}}E(s)(\log D)^{-\Delta}\big) \end{cases}$$

with the Rosser–Iwaniec weight, where the function $E(s)$ is estimated as

$$E(s) = \exp(-s \log s + s \log\log 3s + O(s))$$

for $s \geq 1$ and

$$E(s) = \exp(-s \log s - s \log\log 3s + s \log e\widehat{\kappa})$$

for $1 \leq s \leq (\log z)^{\frac{1}{d}}$ and the constants $\delta > 0$ and $C \geq 1$ depend on κ, Δ, d.

Proof. Recall that

$$V^{\pm}(D, z) = V(z) \pm \sum_{\substack{n \geq 1 \\ n \equiv \nu_{\pm} \pmod 2}} V_n(D, z) = V(z) \pm \lim_{\substack{N \to \infty \\ N \equiv \nu_{\pm} \pmod 2}} T_N(D, z). \tag{14.29}$$

When $1 \leq s \leq (\log z)^{\frac{1}{d}}$, we have

$$\left(1 + \frac{s^d}{\log D}\right)^s \leq \exp\left(\frac{s^{d+1}}{\log D}\right) = \exp\left(\frac{s^d}{\log z}\right) \leq e$$

and so the assertion follows by Lemma 14.4 and (ii) of Proposition 13.1. When $s \geq (\log z)^{\frac{1}{d}}$, i.e. $s \geq (\log D)^{\frac{1}{d+1}}$, we use Lemma 14.3. Since $\omega \in \Omega(\kappa, K)$, we have

$$V(z) \ll K(\log D)^{\kappa} = \exp(O(\log K + \log\log D)) = \exp(O(\log K + \log s))$$

We also have

$$\mathscr{L} \ll \log\log D + \log K \ll \log K + \log 3s = (\log 3s)\left(1 + \frac{\log K}{\log 2s}\right).$$

Therefore, Lemma 14.3 gives

$$\begin{aligned} &T_N(D, z) \\ &\leq V(z)(\log D)^{-\Delta} \\ &\qquad \times \exp\left(-s \log s + s \log\log 3s + O\left(s + \frac{s \log K}{\log 2s} + \log K\right)\right) \\ &\leq e^{\sqrt{K}} V(z)(\log D)^{-\Delta} \exp(-s \log s + s \log\log 3s + O(s)) \end{aligned} \tag{14.30}$$

since if $s \le K^{\frac{1}{3}}$, we have

$$\frac{s\log K}{\log 2s} + \log K \ll K^{\frac{1}{3}} \log K$$

and if $s \ge K^{\frac{1}{3}}$, we have

$$\frac{s\log K}{\log 2s} + \log K \ll s.$$

On inserting (14.30) into (14.29), we obtain the assertion even for this case. □

Theorem 14.8. *Consider*

- *A sieve data* $(\mathcal{A}, \mathcal{P}, z, X, \omega, r)$ *such that* $\omega \in \Omega(\kappa, K)$ *with* $\kappa > 0, K \ge 2$.
- *A level of weight* $D \ge z \ge 2$.
- *A real number* $\Delta \in (0, \Delta_0)$ *with*

$$\Delta_0 = \Delta_0(\kappa) = \begin{cases} 1 & \text{if } \kappa > \frac{1}{2}, \\ \frac{1}{2} & \text{if } 0 < \kappa \le \frac{1}{2} \end{cases}$$

and a real number d *with*

$$d > \frac{7}{\Delta_0 - \Delta}.$$

We then have

$$S(\mathcal{A}, \mathcal{P}, z) \le XV(z)\big(F^+(s) + Ce^{\sqrt{K}}E(s)(\log D)^{-\Delta}\big) + R^+(D, z)$$
$$S(\mathcal{A}, \mathcal{P}, z) \ge XV(z)\big(F^-(s) - Ce^{\sqrt{K}}E(s)(\log D)^{-\Delta}\big) + R^-(D, z)$$

with

$$R^{\pm}(D, z) \coloneqq \sum_{\substack{d<D \\ d|P(z)}} \lambda^{\pm}(d)r(d),$$

where the function $E(s)$ *is estimated as*

$$\begin{cases} E(s) = \exp(-s\log s + s\log\log 3s + O(s)) & \text{for } s \ge 1 \\ E(s) = \exp(-s\log s - s\log\log 3s + s\log e\widehat{\kappa}) & \text{for } 1 \le s \le (\log z)^{\frac{1}{d}} \end{cases}$$

and the constants $\delta > 0$ *and* $C \ge 1$ *depend on* κ, Δ, d.

Proof. It suffices to combine Lemma 3.3 with Lemma 14.7. □

15. Simplest applications to twin prime problem

We check the power of Theorem 14.8 by applying it to twin prime problem.

15.1. **Sieving** $n(n+2)$**.** Let $X \ge 4$ be a real number and let

$$\mathcal{A} \coloneqq \{n(n+2) \mid 1 \le n \le X\}.$$

As the sifting set, we use the set of all primes:

$$\mathcal{P} \coloneqq \{p : \text{prime}\}.$$

For a square-free d, we clearly have

$$|\mathcal{A}_d| = \sum_{\substack{n \le X \\ n(n+2) \equiv 0 \pmod{d}}} 1 = \omega(d)X + r(d),$$

where

$$\omega(d) \coloneqq \frac{\rho(d)}{d}, \quad \rho(d) \coloneqq \#\{x \ (\mathrm{mod}\ d) \mid x(x+2) \equiv 0 \ (\mathrm{mod}\ d)\}$$

and

$$|r(d)| \le \rho(d).$$

By the Chinese remainder theorem, we have

$$\rho(p) = \begin{cases} 1 & \text{if } p = 2, \\ 2 & \text{if } p \ge 3. \end{cases}$$

By Mertens' theorem, for $2 \le w \le z$, we get

$$\begin{aligned} \frac{V(w)}{V(z)} &= \prod_{w \le p < z} \left(1 - \frac{\rho(p)}{p}\right)^{-1} \\ &= \exp\left(\sum_{w \le p < z} \log\left(1 - \frac{\rho(p)}{p}\right)^{-1}\right) \\ &= \exp\left(\sum_{w \le p < z} \frac{\omega(p)}{p} + O\left(\frac{1}{w}\right)\right) \\ &= \left(1 + O\left(\frac{1}{\log w}\right)\right)\left(\frac{\log z}{\log w}\right)^2. \end{aligned}$$

Namely, we have $\omega \in \Omega(\kappa, K)$ with $\kappa = 2$ and a suitable $K \ge 2$. We then apply the lower bound given in Theorem 14.8. To this end, we calculate $F^-(s)$ for small s. Since $\kappa = 2$ now, the parameter β is determined by $\beta = \rho + 1$ with the largest zero ρ of $r_{2,2}(s)$. By Proposition 10.8, we know that

$$r_{2,2}(s) = s^3 - 6s^2 + 9s - \tfrac{8}{3}$$

and so

$$\rho = 3.8339865967\ldots \in (3.8, 3.85) \quad \text{and} \quad \beta \in (4.8, 4.85).$$

By Proposition 11.8 with $\beta = \rho + 1$, we have

$$A > 0 \quad \text{and} \quad B = 0.$$

By Proposition 9.4, we have

$$s^2 T^-(s) = s^2 - 2A \int_\beta^s \frac{t}{(t-1)^2} dt \quad \text{for } \beta \le s \le \beta + 2$$

and so

$$F^-(s) = 1 - T^-(s) = \frac{2A}{s^2} \int_\beta^s \frac{t}{(t-1)^2} dt > 0 \quad \text{for } \beta < s \le \beta + 2.$$

(Since $T^-(s)$ is decreasing, we have $F^-(s) > 0$ for $s > \beta$.) We thus take

$$D = (X+2)^{\frac{485}{490}} \quad \text{and} \quad z = D^{\frac{1}{4.85}} = (X+2)^{\frac{1}{4.90}}$$

which gives

$$s = \frac{\log D}{\log z} = 4.90 \in (\beta, \beta + 1).$$

We also have

$$R^-(D, z) \ll \sum_{d < D} \rho(d)$$

$$\begin{aligned}
&\leq D \sum_{d<D} \omega(d) \\
&\leq D \prod_{p<D} (1-\omega(p))^{-1} \\
&\ll D(\log D)^2 \\
&\ll X^{\frac{485}{490}} (\log X)^2.
\end{aligned}$$

Thus, by Theorem 14.8, we have

$$S(\mathcal{A}, \mathcal{P}, z) \geq cXV(z) - c' X^{\frac{485}{490}} (\log X)^2$$

with some $c, c' > 0$ provided X is sufficiently large. Finally, we have

$$V(z) \gg \left(\frac{V(2)}{V(z)}\right)^{-1} \gg (\log z)^{-2} \gg (\log X)^{-2},$$

we obtain

$$S(\mathcal{A}, \mathcal{P}, z) \gg X(\log X)^{-2}.$$

When $n(n+2)$ is counted in $S(\mathcal{A}, \mathcal{P}, z)$, we have

$$z^{\Omega(n)} \leq n \leq X + 2 \quad \text{and} \quad z^{\Omega(n+2)} \leq n + 2 \leq X + 2$$

and so

$$\max(\Omega(n), \Omega(n+2)) \leq 4.9.$$

Since the left hand side is integer, we get

$$\max(\Omega(n), \Omega(n+2)) \leq 4.$$

Namely, we obtained the following result:

Theorem 15.1. *There are infinitely many pairs* $(n, n+2)$ *of* 4*-almost primes.*

15.2. **Sieving** $p+2$**.** Let $X \geq 4$ be a real number and let

$$\mathcal{A} := \{p + 2 \mid 1 \leq p \leq X\}.$$

As the sifting set, we use the set of all odd primes:

$$\mathcal{P} := \{p : \text{odd prime}\}.$$

For a square-free d, we clearly have

$$|\mathcal{A}_d| = \sum_{\substack{p \leq X \\ p \equiv -2 \pmod{d}}} 1 = \omega(d)\pi(X) + r(X; d, -2),$$

where

$$\omega(d) := \frac{1}{\varphi(d)}.$$

By the Bombieri–Vinogradov theorem, we have

$$\sum_{d<D} |r(X; d, -2)| \ll X(\log X)^{-3}$$

provided

$$D \leq X^{\frac{1}{2}-\varepsilon} \quad \text{for } \varepsilon > 0.$$

By Mertens' theorem, for $2 \le w \le z$, we get

$$\begin{aligned}\frac{V(w)}{V(z)} &= \prod_{\substack{w\le p<z\\ p>2}}\left(1-\frac{1}{\varphi(p)}\right)^{-1}\\ &= \exp\left(\sum_{\substack{w\le p<z\\ p>2}} \log\left(1-\frac{1}{\varphi(p)}\right)^{-1}\right)\\ &= \exp\left(\sum_{\substack{w\le p<z\\ p>2}} \frac{1}{p-1} + O\left(\frac{1}{w}\right)\right)\\ &= \left(1+O\left(\frac{1}{\log w}\right)\right)\left(\frac{\log z}{\log w}\right).\end{aligned}$$

Namely, we have $\omega \in \Omega(\kappa, K)$ with $\kappa = 1$ and a suitable $K \ge 2$. We then apply the lower bound given in Theorem 14.8. To this end, we calculate $F^-(s)$ for small s. Since $\kappa = 2$ now, the parameter β is determined by $\beta = \rho + 1$ with the largest zero ρ of $r_{1,1}(s)$. By Proposition 10.8, we know that

$$r_{1,1}(s) = s - 1$$

and so

$$\rho = 1 \quad \text{and} \quad \beta = 2.$$

By Proposition 11.8 with $\beta = \rho + 1$, we have

$$A > 0 \quad \text{and} \quad B = 0.$$

By Proposition 9.4, we have

$$sT^-(s) = s - A\int_2^s \frac{dt}{t-1} \quad \text{for } 2 \le s \le 4$$

and so

$$F^-(s) = 1 - T^-(s) = \frac{A}{s}\int_2^s \frac{dt}{t-1} > 0 \quad \text{for } 2 < s \le 4.$$

(Since $T^-(s)$ is decreasing, we have $F^-(s) > 0$ for $s > \beta$.) We thus take

$$D = (X+2)^{\frac{21}{43}} \le X^{\frac{1}{2}-\varepsilon} \quad \text{and} \quad z = D^{\frac{1}{2.1}} = (X+2)^{\frac{1}{4.3}}$$

which gives

$$s = \frac{\log D}{\log z} = 2.2 \in (2,3).$$

We also have

$$R^-(D,z) \le \sum_{d<D} |r(X; d, -2)| \ll X(\log X)^{-3}.$$

Thus, by Theorem 14.8, we have

$$S(\mathcal{A}, \mathcal{P}, z) \ge c\pi(X)V(z) - c'X(\log X)^{-3}$$

with some $c, c' > 0$ provided X is sufficiently large. Finally, we have

$$V(z) \gg \left(\frac{V(2)}{V(z)}\right)^{-1} \gg (\log z)^{-1} \gg (\log X)^{-1},$$

we obtain

$$S(\mathcal{A}, \mathcal{P}, z) \gg X(\log X)^{-2}.$$

When $p+2$ is counted in $S(\mathcal{A}, \mathcal{P}, z)$, we have

$$z^{\Omega(p+2)} \leq p+2 \leq X+2$$

and so

$$\Omega(p+2) \leq 4.3.$$

Since the left hand side is integer, we get

$$\Omega(p+2) \leq 4.$$

Namely, we obtained the following result:

Theorem 15.2.
There are infinitely many primes p for which $p+2$ is a 4-almost prime.

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