ON EVEN-ODD AMICABLE PAIRS

Abstract. A pair of positive integers (m, n) is called an amicable pair if the sum of proper divisors of each is equal to the other member. Even though more than 1 billion amicable pairs are known today, it is still unknown if there is any even-odd amicable pair. In this paper, we prove that such even-odd amicable pairs are rare by showing an upper bound for the counting function of even-odd amicable pairs. This improves a remark of Pollack (2011) based on the result of Iannucci and Luca (2009). Our method is a higher-degree version of Pomerance's method (2015), which was used to bound the number of the usual amicable numbers. To extend Pomerance's method to the higher-degree case, for a given positive integer k, we prove an upper bound for the number of smooth values of $\sigma(n^k)$, where $\sigma(n)$ is the sum of divisors of n, and we also prove an upper bound for the number of smooth values of a polynomial and an upper bound for the number of smooth values of a polynomial over primes, which improve the previous results of Hmyrova (1969), Timofeev (1977) and Mine (2024) for a certain range.

1. Introduction

For a positive integer n, let $\sigma(n)$ be the sum of divisors of n and $s(n)$ be the sum of proper divisors of n so that $s(n) = \sigma(n) - n$. A pair of integers (m, n) is called an amicable pair if $s(m) = n$ and $s(n) = m$ or, equivalently, if m and n satisfies the equation

$$\sigma(m) = \sigma(n) = m + n.$$

A member of an amicable pair is called an amicable number. We say that (m, n) is an even-odd amicable pair and m, n are even-odd amicable numbers if (m, n) is an amicable pair consisting of an even number m and an odd number n. Even though more than 1 billion amicable pairs are known today, it is still unknown if there is any even-odd amicable pair. In this paper, we prove that such even-odd amicable pairs are, if any, rare.

In this paper, we measure the "rarity" by bounding the counting functions. The amicable numbers, even if they are not restricted to be even-odd, are known to be rare. Let $A(x)$ be the number of amicable numbers up to x. The first non-trivial upper bound for $A(x)$ is due to Kanold [13], where the upper density of amicable numbers was shown to be less than 0.204. Soon later, Erdős [4] proved the upper density of amicable numbers is 0 and several authors [5, 23, 24, 25, 26] improved the upper bound for $A(x)$. The current best upper bound is

$$A(x) \leq x \exp(-(\tfrac{1}{2} + o(1))(\log x \log\log\log x)^{\frac{1}{2}}) \quad \text{as } x \to \infty \tag{1.1}$$

which is due to Pomerance [25]. Note that the infinitude of amicable pairs is still unknown.

We now get back to the even-odd amicable pairs. Let $B(x)$ be the number of even-odd amicable numbers up to x, i.e. we let

$$B(x) := \#\{m \leq x \mid \text{there exists an amicable pair } (m, n) \text{ with } m \neq n \pmod 2\}.$$

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A simple elementary argument (see Lemma 6.1 at the beginning of Section 6) shows that an even-odd amicable pair (m,n) should be the form

$$(m,n)=(2^aM^2,N^2)\quad\text{with}\quad a\in\mathbb{N}\text{ and }M,N:\text{odd}.$$

At this point, we already know that the even-odd amicable pairs are rare among the whole amicable pairs in the sense that the even-odd amicable pairs form a "quadratic" subsequence of the amicable pairs. This immediately gives the bound

$$B(x)\ll x^{\frac{1}{2}}, \tag{1.2}$$

which we call the trivial bound comparable to $A(x)\le x$. Our next interest is then to obtain upper bounds of $B(x)$ better than the trivial bound (1.2). The only known such bound is a remark of Pollack [22, p. 103], which shows

$$B(x)\le x^{\frac{1}{2}}(\log x)^{-\frac{3}{2}+o(1)}\quad\text{as } x\to\infty \tag{1.3}$$

based on the method of Iannucci and Luca [11]. However, the bound (1.3) has smaller "non-triviality" over (1.2) than (1.1) over $A(x)\le x$.

In this paper, we show the following bound for the even-odd amicable numbers, which has almost the same strength as (1.1):

Theorem 1.1. *We have*

$$B(x)\le x^{\frac{1}{2}}\exp(-(\tfrac{1}{2\sqrt{29}}+o(1))(\log x\log\log\log x)^{\frac{1}{2}})$$

as $x\to\infty$.

Our method is based on Pomerance's argument in [25] but it needs to be extended to the higher degree sequences. As the input of the higher degree version of Pomerance's argument (cf. Lemma 2.1 of [25]), for a given degree k, we need some bounds for the number of squarefree numbers n for which $\sigma(n^k)$ is smooth. For a positive integer n, we let

$$p_{\max}(n)\coloneqq\max\{p:\text{prime factor of } n\}\quad\text{with a convention}\quad p_{\max}(1)=1.$$

For $k\in\mathbb{N}$, $x,y\ge 1$, we let

$$\Sigma_k^\flat(x,y)\coloneqq\#\{n\le x\mid n\colon\text{square-free},\ p_{\max}(\sigma(n^k))\le y\}.$$

We can then prove the following upper bound, which may have its own interest:

Theorem 1.2. *Let* $k\in\mathbb{N}$, $x,y\ge 4$ *and* $u\coloneqq\frac{\log x}{\log y}$. *Then, we have*

$$\Sigma_k^\flat(x,y)\le x\exp(-u\log\log(u+e)-u\log\log(\log(u+1)+e)+o(u))$$

as $u\to\infty$ *in the range*

$$y\ge\log x\log\log x,$$

where the implicit constant depends only on k.

In [25], the result (Lemma 2.1 of [25]) corresponding to Theorem 1.2 was essentially obtained in Banks–Friedlander–Pomerance–Shparlinski [1], whose gap has been corrected in [2]. To prove Theorem 1.2, we use the argument in Section 3 of [2], which was used for large y in the paper [2]. This further demands us, corresponding to Lemma 2.2 on p. 1374 of [2], to bound the number of primes p for which the value of a polynomial $\sigma(p^k)=p^k+\cdots+p+1$ is smooth. More generally, let $F(T)\in\mathbb{Z}[T]$ be a primitive irreducible non-constant polynomial, where a polynomial is primitive if the greatest common divisor of its coefficients is 1, and

$$\pi_F(x,y)\coloneqq\#\{p\le x\mid p_{\max}(F(p))\le y\}.$$

We need to show some upper bound of $\pi_F(x,y)$.

Such upper bounds for $\pi_F(x,y)$ were claimed by Hmyrova [9] or by Timofeev [30]. However, the author could not follow Hmyrova's argument [9] as it is. In particular, Hmyrova's argument for the lower bound of $q_\nu^{(1)}$ at p. 117 of the English translation of [9] seems not working (note that the prime numbers $q_2,\ldots,q_\nu$ in (16) can be much larger than $q_2^{(1)},\ldots,q_\nu^{(1)}$ in (24)) and such an argument imitating Brun's pure sieve is probably insufficient for the purpose there. We can of course try to use a more sophisticated treatment on the multiple sum over primes as in sieve methods, but we do not pursue this direction in this paper. Also, the proof of Lemma 1 of [30, p. 88] seems including a gap that the number $\rho(d)$ of the solutions of a polynomial congruence (mod d) is treated as a complete multiplicative function, which is not true in general. Recently, this issue of Timofeev's lemma has been overcome and the bound itself has been improved for some cases by Mine [15]. Because of such a confusing situation, we reprove the results on smooth values of polynomials in this paper via an alternative method. We also give an upper bound for

$$\Psi_F(x,y) \coloneqq \#\{n \le x \mid p_{\max}(F(n)) \le y\}$$

as well, which was treated in [9, 15, 30]. Our bounds for $\Psi_F(x,y)$ and $\pi_F(x,y)$ are the following. Note that these bounds indeed improve the results in [9, 15, 30] once we have $u\to\infty$ and that the following theorems lose their meaning for fixed u while the results in [9, 15, 30] do.

Theorem 1.3. *Let $F(T)\in\mathbb{Z}[T]$ be a primitive irreducible non-constant polynomial, $x,y\ge 4$ and $u\coloneqq \frac{\log x}{\log y}$. Then, we have*

$$\Psi_F(x,y) \ll x\exp\biggl(-u\log u - u\log\log(u+e) + u - \frac{u\log\log(u+e)}{\log(u+e)} + O\biggl(\frac{u}{\log(u+e)}\biggr)\biggr)$$

provided

$$y \ge \log x \log\log x, \tag{1.4}$$

where the implicit constant depends only on F.

Theorem 1.4. *Let $F(T)\in\mathbb{Z}[T]$ be a primitive irreducible non-constant polynomial, $x,y\ge 4$ and $u\coloneqq \frac{\log x}{\log y}$. Then, we have*

$$\pi_F(x,y) \ll \frac{x}{\log x}\exp\biggl(-u\log u - u\log\log(u+e) + u - \frac{u\log\log(u+e)}{\log(u+e)} + O\biggl(\frac{u}{\log(u+e)}\biggr)\biggr)$$

provided

$$y \ge \log x \log\log x,$$

where the implicit constant depends only on F.

The proofs of Theorem 1.3 and Theorem 1.4 are based on a standard application of Rankin's trick together with an application of an upper bound sieve. This treatment is inspired by Friedlander's argument [6]. The insides of the exponential functions in Theorem 1.3 and Theorem 1.4 coincide with the asymptotic expansion of the Dickman function $\rho(u)$ (see e.g. [3, (1.8), p. 26]) but what we should have here is $\rho(ku)$ with $k=\deg F$ according to Martin [14, (1.4), p. 110]. By allowing a slightly weaker asymptotic formula inside the exponential functions, we can extend the admissible ranges of Theorem 1.2, Theorem 1.3 and Theorem 1.4 to $y\ge\log x$. See Remark 3.1.

This paper is organized as follows. In Section 3, we summarize the outcome of Rankin's trick under a general one-sided setting. In Section 4, we then combine the result of Section 3 with a sieve method to prove Theorem 1.3 and Theorem 1.4. In Section 5, we prove Theorem 1.2. Finally, we prove Theorem 1.1 in Section 6.

2. Notations

In this section, we summarize notations used throughout the paper.

Unless otherwise specified, we use the letters in the following manner: The letters d, e, i, k, m, n, D, M, N, μ, ν denote positive integers, the letters a, b, f denote integers, the letters x, y, z, L denote real numbers usually assumed to be large, the letters u, v, w, z, A, B, U, α, β, ε, κ denote positive real numbers, the letters ℓ, t, σ denote real numbers. The letter p always denotes prime numbers unless otherwise specified. The letter q, P, Q are also used for prime numbers. The letter u denotes a non-negative real number which is usually associated to the real numbers x, y by $u = \frac{\log x}{\log y}$.

For a positive integer n, $\sigma(n)$ denotes the sum of divisors of n, $s(n) := \sigma(n) - n$ denotes the sum of proper divisors of n, $\tau(n)$ denotes the number of divisors of n and $\omega(n)$ denotes the number of distinct prime factors of n. As usual, the arithmetic function μ denotes the Möbius function and φ denotes the Euler totient function. In Section 3, for $\kappa, A > 0$, we use the set $\mathscr{F}(\kappa, A)$ of general multiplicative functions f satisfying the conditions (**F1**), (**F2**) and (**F3**).

For positive integers $m_1, \ldots, m_k$, we write $(m_1, \ldots, m_k)$ for their greatest common divisor, which should be distinguished from a tuple easily by context.

For an integer $n > 1$, we let

$$p_{\max}(n) := \max\{p : \text{prime factor of } n\},$$
$$p_{\min}(n) := \min\{p : \text{prime factor of } n\},$$

and, as a convention, we let $p_{\max}(1) := 1$ and $p_{\min}(1) := \infty$.

Let $F(T) \in \mathbb{Z}[T]$ be a primitive irreducible non-constant polynomial, for which we assume without loss of generality that $F(n) \to \infty$ as $n \to \infty$.

For $x, y \geq 1$, we let

$$\Psi_F(x, y) := \#\{n \leq x \mid p_{\max}(F(n)) \leq y\},$$
$$\pi_F(x, y) := \#\{p \leq x \mid p_{\max}(F(p)) \leq y\},$$
$$\Sigma_k^\flat(x, y) := \#\{n \leq x \mid p_{\max}(\sigma(n^k)) \leq y \text{ and } n\text{: square-free}\}.$$

For $B, \delta > 0$, define functions $g, h\colon [0, \infty) \to \mathbb{R}_{>0}$ given by

$$g(u) = g_B(u) := \exp\left(-u \log u - u \log\log(u+e) + u - \frac{u \log\log(u+e)}{\log(u+e)} + \frac{Bu}{\log(u+e)}\right), \tag{2.1}$$

$$h(u) = h_\delta(u) := \exp(-u \log\log(u+e) - u \log\log(\log(u+1)+e) + \delta u) \tag{2.2}$$

with $g(0) = h(0) = 1$.

If a theorem or a lemma is stated with the phrase "where the implicit constant depends on $a, b, c, \ldots$", then every implicit constant in the corresponding proof may also depend on $a, b, c, \ldots$ without being specifically mentioned.

3. Sums of non-negative arithmetic functions over smooth numbers

In this section, we first prepare some auxiliary estimates for sums of arithmetic functions over smooth numbers. The method used here is just a standard application of Rankin's trick (cf. Section 7.1 of [17, p. 199–215]) but we include proofs for completeness. For possible future use, we prove estimates under a set of general "one-sided" conditions with a relatively wide admissible range by focusing only on the upper bounds. For methods to obtain asymptotic formulas but with narrower admissible ranges, see, e.g. [27, 28, 29].

Throughout this section, we consider a multiplicative function f and real numbers $\kappa, A > 0$. Our conditions on f are the following:

(**F1**) The multiplicative function f is non-negative, i.e. we have $f(n) \geq 0$ for all $n \in \mathbb{N}$.
(**F2**) For real numbers z, w with $2 \leq w \leq z$, we have

$$\sum_{w<p\leq z} \frac{f(p)\log p}{p} \leq \kappa \log \frac{z}{w} + \frac{A}{\log w}.$$

(**F3**) For any prime power p^v with $v \geq 1$, we have $f(p^v) \leq A$.

Let us write $\mathscr{F}(\kappa, A)$ for the set of the multiplicative functions f satisfying (**F1**), (**F2**) and (**F3**).

Remark 3.1. Our density condition (**F2**) is somehow stronger than the usual one, e.g. $(\Omega_2(\kappa))$ of Halberstam–Richert [7]. This is necessary for getting the terms up to u in the exponential function of Theorem 1.3 and Theorem 1.4 and for getting the error term estimate $o(u)$ in Theorem 1.2. We may simplify the arguments in Sections 3 to 5 by using the condition $(\Omega_2(\kappa))$ of Halberstam–Richert and giving up such precise results and satisfied with the term or the error term estimate $O(u)$. This simplification also enables us to have a wider admissible range $y \geq \log x$ as a positive side-effect.

Lemma 3.1. *For $\kappa, A > 0$, $f \in \mathscr{F}(\kappa, A)$ and $x \geq 2$, we have*

$$\sum_{p\leq x} \frac{f(p)}{p} \leq \kappa \log\log x + O(1),$$

where the implicit constant depends only on κ and A.

Proof. This immediately follows by applying partial summation to (**F2**). We have

$$\begin{aligned}\sum_{p\leq x} \frac{f(p)}{p} &= \sum_{p\leq x} \frac{f(p)\log p}{p} \int_p^x \frac{dt}{t(\log t)^2} + \frac{1}{\log x}\sum_{p\leq x} \frac{f(p)\log p}{p} \\ &= \int_2^x \left(\sum_{p\leq t} \frac{f(p)\log p}{p}\right) \frac{dt}{t(\log t)^2} + \frac{1}{\log x}\sum_{p\leq x} \frac{f(p)\log p}{p}.\end{aligned}$$

By (**F2**), we have

$$\begin{aligned}\sum_{p\leq x} \frac{f(p)}{p} &\leq \kappa \int_2^x \frac{dt}{t\log t} + \frac{A}{\log 2}\int_2^x \frac{dt}{t(\log t)^2} + \kappa + \frac{1}{\log x}\frac{A}{\log 2} \\ &\leq \kappa \log\log x + (1 - \log\log 2)\kappa + \frac{A}{(\log 2)^2}.\end{aligned}$$

This completes the proof. □

Lemma 3.2. *For $\kappa, A > 0$, $f \in \mathscr{F}(\kappa, A)$, $y \geq 2$ and $0 \leq \sigma < 1$, we have*

$$\sum_{p\leq y} \frac{f(p)}{p^\sigma} \leq \kappa \log \frac{1}{1-\sigma} + \frac{\kappa y^{1-\sigma}}{(1-\sigma)\log y} + O\left(\frac{y^{1-\sigma}}{((1-\sigma)\log y)^2}\right),$$

where the implicit constant depends only on κ and A.

Proof. We first remark that the Taylor expansion of the exponential function implies

$$\frac{y^{1-\sigma}}{((1-\sigma)\log y)^2} = \frac{\exp((1-\sigma)\log y)}{((1-\sigma)\log y)^2} \geq \frac{1+(1-\sigma)\log y + \frac{1}{2}((1-\sigma)\log y)^2 + \cdots}{((1-\sigma)\log y)^2} \geq \frac{1}{2} \tag{3.1}$$

so any error term of the size $O(1)$ is negligible below.

Let

$$y_1 := \exp\biggl(\frac{1}{1-\sigma}\biggr).$$

We first decompose the sum as

$$\sum_{p\leq y}\frac{f(p)}{p^\sigma} \leq \sum_{p\leq y_1}\frac{f(p)}{p^\sigma} + \sum_{y_1<p\leq y}\frac{f(p)}{p^\sigma} =: \sum\nolimits_1 + \sum\nolimits_2, \quad \text{say.} \tag{3.2}$$

(The first inequality cannot be equality when $y < y_1$.)

We then consider the sum $\sum_1$. Since

$$\frac{d}{dt}\biggl(\frac{t^{1-\sigma}}{\log t}\biggr) = \frac{t^{-\sigma}}{\log t}\biggl((1-\sigma) - \frac{1}{\log t}\biggr) \leq 0 \quad \text{for} \quad t \leq y_1,$$

by using (**F2**), we have

$$\begin{aligned}\sum\nolimits_1 &= \sum_{p\leq y_1}\frac{f(p)\log p}{p}\int_p^{y_1}\frac{d}{dt}\biggl(-\frac{t^{1-\sigma}}{\log t}\biggr)dt + \frac{y_1^{1-\sigma}}{\log y_1}\sum_{p\leq y_1}\frac{f(p)\log p}{p}\\ &= \int_2^{y_1}\biggl(\sum_{p\leq t}\frac{f(p)\log p}{p}\biggr)\frac{d}{dt}\biggl(-\frac{t^{1-\sigma}}{\log t}\biggr)dt + \frac{y_1^{1-\sigma}}{\log y_1}\sum_{p\leq y_1}\frac{f(p)\log p}{p}\\ &\leq \int_2^{y_1}\biggl(\kappa\log t + \frac{A}{\log 2}\biggr)\frac{d}{dt}\biggl(-\frac{t^{1-\sigma}}{\log t}\biggr)dt + \frac{y_1^{1-\sigma}}{\log y_1}\biggl(\kappa\log y_1 + \frac{A}{\log 2}\biggr)\\ &= \kappa\int_2^{y_1}\frac{dt}{t^\sigma\log t} + O(1),\end{aligned} \tag{3.3}$$

where we used (**F2**) in the last inequality. For the last integral, since

$$2 \leq t \leq y_1 \implies t^{1-\sigma} = \exp((1-\sigma)\log t) = 1 + O((1-\sigma)\log t)$$

we have

$$\kappa\int_2^{y_1}\frac{dt}{t^\sigma\log t} = \kappa\int_2^{y_1}\frac{dt}{t\log t} + O\biggl((1-\sigma)\int_2^{y_1}\frac{dt}{t}\biggr) = \log\frac{1}{1-\sigma} + O(1).$$

By substituting this formula into (3.3), we obtain

$$\sum\nolimits_1 \leq \kappa\log\frac{1}{1-\sigma} + O(1). \tag{3.4}$$

We next consider the sum $\sum_2$. Since

$$\frac{d}{dt}\biggl(\frac{t^{1-\sigma}}{\log t}\biggr) = \frac{t^{-\sigma}}{\log t}\biggl((1-\sigma) - \frac{1}{\log t}\biggr) \geq 0 \quad \text{for} \quad t > y_1,$$

by using (**F2**), we have

$$\sum\nolimits_2 = \sum_{y_1<p\leq y}\frac{f(p)\log p}{p}\int_{y_1}^{p}\frac{d}{dt}\biggl(\frac{t^{1-\sigma}}{\log t}\biggr)dt + \frac{y_1^{1-\sigma}}{\log y_1}\sum_{y_1<p\leq y}\frac{f(p)\log p}{p}$$

$$
\begin{aligned}
&= \int_{y_1}^{y} \bigg(\sum_{t<p\le y} \frac{f(p)\log p}{p} \bigg) \frac{d}{dt}\bigg(\frac{t^{1-\sigma}}{\log t} \bigg) dt + \frac{y_1^{1-\sigma}}{\log y_1} \sum_{y_1<p\le y} \frac{f(p)\log p}{p} \\
&\le \int_{y_1}^{y} \bigg(\kappa \log\frac{y}{t} + \frac{A}{\log t} \bigg) \frac{d}{dt}\bigg(\frac{t^{1-\sigma}}{\log t} \bigg) dt + \frac{y_1^{1-\sigma}}{\log y_1} \bigg(\kappa\log\frac{y}{y_1} + \frac{A}{\log y_1} \bigg) \\
&= \kappa \int_{y_1}^{y} \frac{dt}{t^{\sigma}\log t} + O\bigg(\frac{y^{1-\sigma}}{(\log y)^2} \bigg)
\end{aligned}
$$

where we used (**F2**) in the last inequality. For the last integral, since

$$
\int_{y_1}^{y_1^4} \frac{dt}{t^{\sigma}\log t} = \int_{y_1}^{y_1^4} \frac{t^{1-\sigma}dt}{t\log t} \le e^4 \int_{y_1}^{y_1^4} \frac{dt}{t\log t} = \log 4,
$$

by using the integration by parts, we can further continue the above estimate to obtain

$$
\begin{aligned}
\textstyle\sum_2 &\le \kappa \int_{y_1^4}^{y} \frac{dt}{t^{\sigma}\log t} + O\bigg(\frac{y^{1-\sigma}}{((1-\sigma)\log y)^2} \bigg) \\
&\le \frac{\kappa y^{1-\sigma}}{(1-\sigma)\log y} + \frac{\kappa}{1-\sigma} \int_{y_1^4}^{y} \frac{dt}{t^{\sigma}(\log t)^2} + O\bigg(\frac{y^{1-\sigma}}{((1-\sigma)\log y)^2} \bigg) \\
&=: \frac{\kappa y^{1-\sigma}}{(1-\sigma)\log y} + \kappa I + O\bigg(\frac{y^{1-\sigma}}{((1-\sigma)\log y)^2} \bigg), \quad \text{say,}
\end{aligned} \tag{3.5}
$$

by using (3.1). By integration by parts, we have

$$
\begin{aligned}
I &\le \frac{y^{1-\sigma}}{((1-\sigma)\log y)^2} + \frac{2}{(1-\sigma)^2} \int_{y_1^4}^{y} \frac{dt}{t^{\sigma}(\log t)^3} \\
&\le \frac{y^{1-\sigma}}{((1-\sigma)\log y)^2} + \frac{1}{2(1-\sigma)} \int_{y_1^4}^{y} \frac{dt}{t^{\sigma}(\log t)^2} = \frac{y^{1-\sigma}}{((1-\sigma)\log y)^2} + \frac{1}{2}I
\end{aligned}
$$

so that

$$
I \ll \frac{y^{1-\sigma}}{((1-\sigma)\log y)^2}.
$$

By substituting this formula into (3.5), we obtain

$$
\textstyle\sum_2 = \frac{\kappa y^{1-\sigma}}{(1-\sigma)\log y} + O\bigg(\frac{y^{1-\sigma}}{((1-\sigma)\log y)^2} \bigg) \tag{3.6}
$$

On inserting (3.4) and (3.6) into (3.2), we obtain the lemma. □

Recall the definition of the function $g = g_B$ given in (2.1).

Lemma 3.3. *Let $\kappa, A > 0$, $f \in \mathscr{F}(\kappa, A)$, $x \ge 1$, $y \ge 2$ and $u := \frac{\log x}{\log y}$. Then, we have*

$$
\sum_{\substack{n>x \\ p_{\max}(n)\le y}} \frac{f(n)}{n} \ll (\log y)^{\kappa} g_B(u) \kappa^u
$$

provided

$$
y \ge \log x \log\log x, \tag{3.7}
$$

where the constant $B > 0$ and the implicit constant depend only on κ and A.

Proof. By (**F1**), (**F3**) and Lemma 3.1, we have a trivial bound

$$\sum_{\substack{n>x \\ p_{\max}(n)\le y}} \frac{f(n)}{n} \le \prod_{p\le y}\left(1+\sum_{v=1}^{\infty}\frac{f(p^v)}{p^v}\right) \le \exp\left(\sum_{p\le y}\sum_{v=1}^{\infty}\frac{f(p^v)}{p^v}\right)$$
$$\le \exp\left(\sum_{p\le y}\frac{f(p)}{p}+\sum_{p}\frac{A}{p(p-1)}\right) \ll (\log y)^{\kappa}$$

Thus, by (3.7), we may assume x, y, u are sufficiently large.

Let

$$\sigma = 1-\frac{\log\frac{u}{\kappa}}{\log y}-\frac{\log\log(u+e)}{\log y} = 1-\frac{\log(u\log(u+e))}{\log y}+\frac{\log\kappa}{\log y}. \tag{3.8}$$

By (3.7), we have $u \le \log x - e$ and so

$$u\log(u+e) \le \frac{\log x}{\log y}\log\log x \le \log x$$

for large y. Thus, by (3.8), we have

$$1-\frac{\log\log x}{\log y}+\frac{\log\kappa}{\log y} \le \sigma \le 1-\frac{\log u}{\log y} \le 1-\frac{1}{\log y} \tag{3.9}$$

for large y, u. By using Rankin's trick, (**F1**) and (**F3**),

$$\begin{aligned}\sum_{\substack{n>x \\ p_{\max}(n)\le y}} \frac{f(n)}{n} &\le x^{\sigma-1}\sum_{p_{\max}(n)\le y}\frac{f(n)}{n^{\sigma}} \\ &\le x^{\sigma-1}\prod_{p\le y}\left(1+\sum_{v=1}^{\infty}\frac{f(p^v)}{p^{v\sigma}}\right) \\ &\le x^{\sigma-1}\exp\left(\sum_{p\le y}\sum_{v=1}^{\infty}\frac{f(p^v)}{p^{v\sigma}}\right) \\ &\le x^{\sigma-1}\exp\left(\sum_{p\le y}\frac{f(p)}{p^{\sigma}}+O\left(\sum_{p\le y}\frac{1}{p^{\sigma}(p^{\sigma}-1)}\right)\right).\end{aligned} \tag{3.10}$$

By (3.9), we have $0 \le \sigma < 1$ for large y. Thus, by Lemma 3.2, (3.8) and (3.9), we have

$$\begin{aligned}\sum_{p\le y}\frac{f(p)}{p^{\sigma}} &\le \kappa\log\frac{1}{1-\sigma}+\frac{\kappa y^{1-\sigma}}{(1-\sigma)\log y}+O\left(\frac{y^{1-\sigma}}{((1-\sigma)\log y)^2}\right) \\ &\le \kappa\log\log y+\frac{u\log(u+e)}{\log u+\log\log(u+e)+\log\kappa}+O\left(\frac{u\log(u+e)}{(\log(u\log(u+e)))^2}\right) \\ &= \kappa\log\log y+u-\frac{u\log\log(u+e)}{\log u}+O\left(\frac{u}{\log u}\right)\end{aligned}$$

for large u. By substituting this estimate into (3.10) and recalling (3.8), we have

$$\sum_{\substack{n>x \\ p_{\max}(n)\le y}} \frac{f(n)}{n} \le (\log y)^{\kappa}\kappa^{u}\exp\Bigg(-u\log u - u\log\log(u+e)+u$$

$$- \frac{u \log\log(u+e)}{\log u} + O\biggl(\frac{u}{\log u} + \sum_{p \le y} \frac{1}{p^\sigma(p^\sigma - 1)}\biggr)\biggr).$$

Thus, it suffices to prove

$$\sum_{p \le y} \frac{1}{p^\sigma(p^\sigma - 1)} \ll \frac{u}{\log u}. \tag{3.11}$$

We consider three cases separately.

We first consider the case $y \ge (\log x)^3$. In this case, by (3.9), we have

$$\sigma \ge 1 - \frac{\log\log x}{\log y} + \frac{\log \kappa}{\log y} \ge \frac{2}{3} + \frac{\log \kappa}{\log y}.$$

Therefore, for large y, we have $p^\sigma - 1 \gg p^\sigma$ and so

$$\sum_{p \le y} \frac{1}{p^\sigma(p^\sigma - 1)} \ll \sum_{p \le y} \frac{1}{p^{2\sigma}} \ll \sum_{p} \frac{1}{p^{\frac{4}{3}}} < +\infty.$$

Thus, (3.11) holds for the case $y \ge (\log x)^3$.

We next consider the case $(\log x)^{\frac{8}{5}} \le y \le (\log x)^3$. In this case, by (3.9), we have

$$\sigma \ge 1 - \frac{\log\log x}{\log y} + \frac{\log \kappa}{\log y} \ge \frac{3}{8} + \frac{\log \kappa}{\log y}.$$

Thus, for large y, we have $p^\sigma - 1 \gg p^\sigma$ and so

$$\sum_{p \le y} \frac{1}{p^\sigma(p^\sigma - 1)} \ll \sum_{p \le y} \frac{1}{p^{2\sigma}} \ll \sum_{p \le y} \frac{1}{p^{\frac{3}{4}}} \ll y^{\frac{1}{4}} \le (\log x)^{\frac{3}{4}} \ll \frac{\log x}{(\log\log x)^2} \ll \frac{u}{\log u}.$$

Thus, (3.11) holds for the case $(\log x)^{\frac{5}{8}} \le y \le (\log x)^3$.

We finally consider the case $\log x \log\log x \le y \le (\log x)^{\frac{5}{8}}$. We decompose the sum as

$$\sum_{p \le y} \frac{1}{p^\sigma(p^\sigma - 1)} = \sum_{p \le e^{\frac{1}{\sigma}}} + \sum_{e^{\frac{1}{\sigma}} < p \le y} = \sum\nolimits_1 + \sum\nolimits_2. \tag{3.12}$$

We first estimate the sum $\sum_1$. By (3.9), we have

$$\sigma \ge 1 - \frac{\log \frac{\log x}{\kappa}}{\log y} \ge 1 - \frac{\log \frac{\log x}{\kappa}}{\log(\log x \log\log x)} = \frac{\log\log\log x + \log \kappa}{\log\log x + \log\log\log x} \ge \frac{2}{\log\log x}$$

for large x. Thus, by using $p^\sigma - 1 \ge \sigma \log p$ and Chebyshev's bound, we get

$$\sum\nolimits_1 \le \frac{1}{\sigma} \sum_{p \le e^{\frac{1}{\sigma}}} \frac{1}{\log p} \ll \sigma e^{\frac{1}{\sigma}} \ll (\log x)^{\frac{1}{2}} \ll \frac{\log x}{(\log\log x)^2} \ll \frac{u}{\log u}. \tag{3.13}$$

In the sum $\sum_2$, we can use $p^\sigma - 1 = p^\sigma(1 - p^{-\sigma}) \ge p^\sigma(1 - e^{-1})$ to get

$$\sum\nolimits_2 \ll \sum_{p \le y} \frac{1}{p^{2\sigma}}. \tag{3.14}$$

By (3.9), we have

$$\sigma \le 1 - \frac{\log\log x}{\log y} + \frac{\log\log y}{\log y} \le \frac{3}{8} + \frac{\log\log y}{\log y} \le \frac{4}{9} \quad \text{so that} \quad 2\sigma \in [0, 1)$$

for large y. Thus, by applying Lemma 3.2 to (3.14), we have

$$\sum\nolimits_2 \ll \log \frac{1}{1-2\sigma} + \frac{y^{1-2\sigma}}{(1-2\sigma)\log y} \ll \frac{y^{1-2\sigma}}{\log y}$$

where we used $(1-2\sigma)\log y \geq \frac{1}{9}\log y \geq 1$ for large y. On inserting (3.8), this gives

$$\begin{aligned}\sum\nolimits_2 \ll \frac{y^{2(1-\sigma)}}{y\log y} = \frac{(u\log(u+e))^2}{y\log y} &= \frac{u}{\log(u+e)}\frac{u(\log(u+e))^3}{y\log y} \\ &\ll \frac{u}{\log(u+e)}\frac{\log x(\log\log x)^3}{y(\log y)^2} \ll \frac{u}{\log(u+e)}.\end{aligned} \tag{3.15}$$

On inserting (3.13) and (3.15) into (3.12), we obtain (3.11) for the last case $\log x \log\log x \leq y \leq (\log x)^{\frac{5}{8}}$. This completes the proof. □

Lemma 3.4. *For $B > 0$, we have*

$$g(u-t) \leq g(u)(e^3 u\log(u+e))^t \qquad \text{for } u > 0 \text{ and } u \geq t \geq 0$$

and

$$g(u+t) \leq g(u)(e^{-\frac{B}{\log(u+e)}} u\log(u+e))^{-t} \quad \text{for } u > 0 \text{ and } t \geq 0.$$

Proof. Let

$$\ell(u) := -u\log u - u\log\log(u+e) + u - \frac{u\log\log(u+e)}{\log(u+e)} + \frac{Bu}{\log(u+e)}$$

so that $g(u) = \exp(\ell(u))$. Since

$$\begin{aligned}\ell'(u) = &-\log(u\log(u+e)) - \frac{u}{(u+e)\log(u+e)} \\ &- \frac{\log\log(u+e)}{\log(u+e)} + \frac{u\log\log(u+e)}{(u+e)(\log(u+e))^2} - \frac{u}{(u+e)(\log(u+e))^2} \\ &+ B\left(\frac{1}{\log(u+e)} - \frac{u}{(u+e)(\log(u+e))^2}\right),\end{aligned}$$

we have

$$-\log(e^3 u\log(u+e)) \leq \ell'(u) \leq -\log(e^{-\frac{B}{\log(u+e)}} u\log(u+e)).$$

Therefore, by the mean value theorem, we have

$$\begin{aligned}\ell(u-t) = \ell(u) + \ell'(\xi)(-t) &\leq \ell(u) + (-\log(e^3\xi\log(\xi+e)))(-t) \\ &\leq \ell(u) + (\log(e^3 u\log(u+e)))t\end{aligned}$$

where $\xi \in [u-t, u]$, and

$$\begin{aligned}\ell(u+t) = \ell(u) + \ell'(\xi)t &\leq \ell(u) + (-\log(e^{-\frac{B}{\log(\xi+e)}}\xi\log(\xi+e)))t \\ &\leq \ell(u) + (-\log(e^{-\frac{B}{\log(u+e)}} u\log(u+e)))t\end{aligned}$$

where $\xi \in [u, u+t]$. By exponentiating these inequalities, we obtain the lemma. □

Lemma 3.5. *Let $\kappa, A > 0$, $f \in \mathscr{F}(\kappa, A)$, $x \geq 1$, $y \geq 2$, q be a prime and $u := \frac{\log x}{\log y}$. Then,*

$$\sum_{\substack{n>x \\ q|n \\ p_{\max}(n)\leq y}} \frac{f(n)}{n} \ll \frac{(\log y)^\kappa}{q} g_B(u)\kappa^u$$

provided

$$y \geq \log x \log\log x \quad \text{and} \quad q \leq y, \tag{3.16}$$

where the constant $B > 0$ and the implicit constant depend only on κ and A.

Proof. By (**F1**), (**F3**) and Lemma 3.3,

$$\sum_{\substack{n>x \\ q|n \\ p_{\max}(n)\leq y}} \frac{f(n)}{n} = \sum_{m=1}^{\infty} \sum_{\substack{n>x \\ q^m\|n \\ p_{\max}(n/q^m)\leq y}} \frac{f(n)}{n} \leq \left(\sum_{m=1}^{\infty} \frac{f(q^m)}{q^m}\right) \sum_{p_{\max}(n)\leq y} \frac{f(n)}{n} \ll \frac{(\log y)^\kappa}{q}.$$

Hence, we may assume that x, y, u are sufficiently large. In particular, we may assume $y \leq x$. Let $g(u) = g_{B_1}(u)$ as defined in (2.1) with sufficiently large constant $B_1 > 0$ depending on κ and A.

We decompose the sum as

$$\sum_{\substack{n>x \\ q|n \\ p_{\max}(n)\leq y}} \frac{f(n)}{n} = \sum_{m=1}^{\infty} \sum_{\substack{n>x \\ q^m\|n \\ p_{\max}(n/q^m)\leq y}} \frac{f(n)}{n} = \sum_{q^m\leq x} + \sum_{q^m>x} = \sum\nolimits_1 + \sum\nolimits_2, \quad \text{say.} \tag{3.17}$$

For the sum $\sum_1$, by (**F1**), (**F3**), we obtain

$$\sum\nolimits_1 \ll \sum_{q^m\leq x} \frac{1}{q^m} \sum_{\substack{n>x/q^m \\ p_{\max}(n)\leq y}} \frac{f(n)}{n}.$$

By noting that

$$y \geq \log x \log\log x \geq \log \frac{x}{q^m} \log\log \frac{x}{q^m} \quad \text{and} \quad \frac{x}{q^m} \geq 1$$

and using Lemma 3.3 and Lemma 3.4, we have

$$\begin{aligned}
\sum\nolimits_1 &\ll \sum_{q^m\leq x} \frac{(\log y)^\kappa}{q^m} g\left(u - m\frac{\log q}{\log y}\right) \kappa^{u - m\frac{\log q}{\log y}} \\
&\ll g(u)\kappa^u (\log y)^\kappa \sum_{m=1}^{\infty} \left(\frac{\kappa e^3 u \log(u+e)}{y}\right)^{m\frac{\log q}{\log y}} \\
&\ll g(u)\kappa^u (\log y)^\kappa \left(\frac{\kappa e^3 u \log(u+e)}{y}\right)^{\frac{\log q}{\log y}} \left(1 - \left(\frac{\kappa e^3 u \log(u+e)}{y}\right)^{\frac{\log q}{\log y}}\right)^{-1}
\end{aligned}$$

since the assumption (3.16) implies

$$\frac{\kappa e^3 u \log(u+e)}{y} \leq \frac{\kappa e^3 \log x \log\log x}{y \log y} \leq \frac{\kappa e^3}{\log y} \leq \frac{1}{e} \tag{3.18}$$

for large y. When $y \geq (\log x)^2$, we have

$$u\log(u+e) \leq \frac{y^{\frac{1}{2}}}{\log y}\log\left(\frac{y^{\frac{1}{2}}}{\log y}+e\right) \leq y^{\frac{1}{2}} \leq \kappa^{-1}e^{-4}y^{\frac{2}{3}}$$

for large y so that

$$\left(1-\left(\frac{\kappa e^3 u\log(u+e)}{y}\right)^{\frac{\log q}{\log y}}\right)^{-1} \leq (1-q^{-\frac{1}{3}})^{-1} \ll 1.$$

When $\log x \leq y \leq (\log x)^2$, we have

$$u = \frac{\log x}{\log y} \geq \frac{y^{\frac{1}{2}}}{\log y} \geq \log y \tag{3.19}$$

for large y so that by recalling (3.16) and (3.18), we have

$$\left(1-\left(\frac{\kappa e^3 u\log(u+e)}{y}\right)^{\frac{\log q}{\log y}}\right)^{-1} \leq (1-e^{-\frac{\log q}{\log y}})^{-1} \ll \frac{\log y}{\log q} \ll u = \exp(O(\log u)).$$

Thus, in any case, by using (3.16), we have

$$\begin{aligned}\sum\nolimits_1 &\leq g(u)\kappa^u(\log y)^\kappa\left(\frac{u\log(u+e)}{ey}\right)^{\frac{\log q}{\log y}}\exp(O(\log u))\\ &= g(u)\kappa^u\frac{(\log y)^\kappa}{q}\left(\frac{u\log(u+e)}{e}\right)^{\frac{\log q}{\log y}}\exp(O(\log u)) \leq \frac{(\log y)^\kappa}{q}g(u)\kappa^u\exp(O(\log u)).\end{aligned} \tag{3.20}$$

We next consider the sum $\sum_2$. When $y \geq (\log x)^2$, we have

$$\begin{aligned}\frac{1}{g(u)\kappa^u} &\leq \exp(u\log u + u\log\log(u+e) - u\log\kappa)\\ &\leq \exp\left(\frac{3}{2}u\log\log x\right) \leq \exp\left(\frac{3}{4}\frac{\log x}{\log\log x}\log\log x\right) \leq x^{\frac{3}{4}}\end{aligned}$$

for large x so that by (**F1**), (**F3**), Lemma 3.3 and (3.16),

$$\sum\nolimits_2 \ll \sum_{q^m>x}\frac{1}{q^m}\sum_{p_{\max}(n)\leq y}\frac{f(n)}{n} \ll (\log y)^\kappa\sum_{q^m>x}\frac{1}{q^m} \ll \frac{(\log y)^\kappa}{x} = \frac{(\log y)^\kappa}{y^{\frac{u}{4}}x^{\frac{3}{4}}} \leq \frac{(\log y)^\kappa}{q}g(u)\kappa^u$$

provided $u \geq 4$. On the other hand, when $\log x\log\log x \leq y \leq (\log x)^2$, we have

$$\frac{1}{g(u)\kappa^u} \leq \exp(u\log u + u\log\log(u+e) - u\log\kappa) \leq \exp\left(\log x\cdot\frac{\log\log x - \log\kappa}{\log\log x + \log\log\log x}\right) \leq x$$

and (3.19) for large x, u so that by (**F3**), Lemma 3.3 and (3.16),

$$\sum\nolimits_2 \ll (\log y)^\kappa\sum_{q^m>x}\frac{1}{q^m} \ll \frac{(\log y)^\kappa}{x} = \frac{(\log y)^\kappa}{y}\frac{1}{x}\exp(\log y) \leq \frac{(\log y)^\kappa}{q}g(u)\kappa^u\exp(O(\log u))$$

for large u. Thus, in any case,

$$\sum\nolimits_2 \leq \frac{(\log y)^\kappa}{q}g(u)\kappa^u\exp(O(\log u)). \tag{3.21}$$

On inserting (3.20) and (3.21) into (3.17), we obtain the lemma. □

Lemma 3.6. *Let $\kappa, A > 0$, $f \in \mathscr{F}(\kappa, A)$, $x \geq 1$, q be a prime and $u := \frac{\log x}{\log q}$. Then,*

$$\sum_{\substack{n>x \\ p_{\max}(n)=q}} \frac{f(n)}{n} \ll \frac{(\log q)^\kappa}{q} g_B(u)\kappa^u$$

provided

$$q \geq \log x \log\log x,$$

where the constant $B > 0$ and the implicit constant depend only on κ and A.

Proof. This follows just by taking $y = q$ in Lemma 3.5. □

4. Smooth values of a polynomial over prime numbers

In this section, we shall prove Theorem 1.3 and Theorem 1.4.

4.1. **Auxiliary lemmas on polynomial congruences.**

Throughout this section, we let

$F(T) \in \mathbb{Z}[T]$ be an irreducible non-constant polynomial with coprime coefficients.

Note that the primitivity, i.e. the assumption that the greatest common divisor of the coefficients of F is 1 makes no loss of generality for the proof of neither of Theorem 1.3 nor Theorem 1.4. Also, we may multiply F by -1 if necessary to assume that the leading coefficient of F is positive. Let

$$k := \deg F.$$

Also, for $\ell \in \mathbb{R}_{\geq 0}$, we use arithmetic functions

$$r_F(n) := \#\{x \ (\mathrm{mod}\ n) \mid F(x) \equiv 0 \ (\mathrm{mod}\ n)\} \quad \text{and} \quad r_{F,\ell} := r_F(n)\left(\frac{n}{\varphi(n)}\right)^\ell.$$

We first collect some basic properties of $r_{F,\ell}$.

Lemma 4.1. *For the arithmetic function $r_{F,\ell}$, we have:*

(i) *The arithmetic function $r_{F,\ell}$ is multiplicative.*
(ii) *We have $0 \leq r_{F,\ell}(p^v) \ll_{F,\ell} 1$ for prime powers p^v.*
(iii) *There exists a real constant c_F such that*

$$\sum_{p \leq x} \frac{r_{F,\ell}(p)\log p}{p} = \log x + c_F + O\left(\frac{1}{\log x}\right)$$

for $x \geq 2$, where the implicit constant depends only on F, ℓ.

(iv) *We have $r_{F,\ell} \in \mathscr{F}(1, A)$ with some $A \geq 0$ depending only on F and ℓ.*

Proof.

(i). The arithmetic function r_F is multiplicative by the Chinese remainder theorem and the Euler totient function φ is well-known to be multiplicative. Thus, the assertion is clear.

(ii). The lower bound is trivial. For the upper bound, we first prove $r_F(p^v) \ll_F 1$. Since $\mathbb{Z}/p\mathbb{Z}$ is a field and F is primitive, we have $r_F(p) \leq k$. Then, for general prime power p^v, we can trace the Hensel lifting with taking care of the effect of the discriminant of F, which is non-zero since F is irreducible. For details, see e.g. Théorème II of Nagel [18, p. 349] or Theorem 54 of Nagell's book [19, p. 90]. See also [10] for a stronger estimate. Then, the assertion follows by $\frac{p^v}{\varphi(p^v)} = \frac{p}{p-1} \leq 2$.

(iii). Let θ be a root of the polynomial F. When F is monic, by the Dedekind–Kummer theorem (see, e.g. Proposition 8.3 on p. 47–48 or Exercise 2 on p. 52 of Neukirch [21]) and the sparseness of prime ideals of the inertia degree ≥ 2, the prime ideal theorem (see e.g. Corollary 1.(ii) of Theorem 7.20 of [20, p. 358]) in the number field $\mathbb{Q}(\theta)$ implies

$$\sum_{p\leq x}\frac{r_F(p)\log p}{p} = \log x + c_F + O\left(\frac{1}{\log x}\right)$$

(cf. Section 2 of [18, p. 349–352]). Even if F is not monic, we can use the same argument by multiplying the $(k-1)$-th power of the leading coefficient of F to F itself. For details, see the second paragraph of p. 352 of Nagel [18]. Then, the assertion follows by $\frac{p}{\varphi(p)} = 1 + O(\frac{1}{p})$.

(iv). This follows by (i), (ii) and (iii) proven above. □

Lemma 4.2. *Let $x \geq 1$, $y \geq 2$ and $u \coloneqq \frac{\log x}{\log y}$. Then, we have*

$$\sum_{\substack{n>x\\ p_{\max}(n)\leq y}}\frac{r_{F,\ell}(n)}{n} \ll (\log y) g_B(u)$$

provided

$$y \geq \log x \log\log x,$$

where the constant $B > 0$ and the implicit constant depend only on F and ℓ.

Proof. This follows by (iv) of Lemma 4.1 and Lemma 3.3 with $\kappa = 1$. □

Lemma 4.3. *Let $x \geq 1$, q be a prime number and $u \coloneqq \frac{\log x}{\log q}$. Then, we have*

$$\sum_{\substack{n>x\\ p_{\max}(n)=q}}\frac{r_{F,\ell}(n)}{n} \ll \frac{\log q}{q} g_B(u)$$

provided

$$q \geq \log x \log\log x$$

where the constant $B > 0$ and the implicit depend only on F and ℓ.

Proof. This follows by (iv) of Lemma 4.1 and Lemma 3.6 with $\kappa = 1$. □

4.2. **Auxiliary lemmas for the application of a sieve method.** Our treatment of $\Psi_F(x, y)$ and $\pi_F(x, y)$ requires some sieve method. We thus prepare some auxiliary results from sieve theory. For convenience in the error term estimate, we shall use the arithmetic large sieve.

Lemma 4.4 (Arithmetic large sieve)**.** *Let x_0, x, y be real numbers with $x, y \geq 1$. Suppose that a set $\Omega(p)$ of residues $(\mathrm{mod}\ p)$ is given for each prime $p \leq y$. Let $\omega(p) \coloneqq \#\Omega(p)$. Then,*

$$\#\{n \in (x_0, x_0 + x] \cap \mathbb{Z} \mid n\ (\mathrm{mod}\ p) \notin \Omega(p) \text{ for all } p \leq y\} \ll \frac{x + y^2}{G(y)},$$

where

$$G(y) \coloneqq \sum_{d\leq y}\mu(d)^2\prod_{p\mid d}\frac{\omega(p)}{p-\omega(p)}$$

and the implicit constant is absolute.

Proof. See Corollary 3.2 of [16, p. 25–26]. □

Lemma 4.5. *Let y, κ, A be a real number with $y \geq 2$, $\kappa > 0$ and $A \geq 1$. Suppose that a real number $\omega(p)$ is given for each prime $p \leq y$ and satisfies*

$$0 \leq \frac{\omega(p)}{p} \leq 1 - \frac{1}{A} \quad \text{for all} \quad p \leq y \tag{Ω_1}$$

$$\sum_{w<p\leq z} \frac{\omega(p)\log p}{p} \leq \kappa \log \frac{z}{w} + A \quad \text{for all} \quad 2 \leq w \leq z \leq y. \tag{$\Omega_2(\kappa)$}$$

Then, for the sum $G(y)$ defined as in Lemma 4.4, we have

$$\frac{1}{G(y)} \ll \prod_{p\leq y}\left(1 - \frac{\omega(p)}{p}\right),$$

where the implicit constant depends only on κ and A.

Proof. Take $x \coloneqq y$ and $z \coloneqq y$ in Lemma 4.1 of [7, p. 131]. □

Lemma 4.6. *For $x \geq 2$, we have*

$$\prod_{p\leq x}\left(1 - \frac{r_F(p)}{p}\right) \ll \frac{1}{\log x},$$

where the implicit constant depends only on F.

Proof. This is just an analog of Mertens' theorem. By (ii) of Lemma 4.1, we obtain

$$\begin{aligned}\prod_{p\leq x}\left(1 - \frac{r_F(p)}{p}\right) &= \exp\left(\sum_{p\leq x}\log\left(1 - \frac{r_F(p)}{p}\right)\right)\\ &= \exp\left(-\sum_{p\leq x}\frac{r_F(p)}{p} + O\left(\sum_{p\leq x}\frac{r_F(p)^2}{p^2}\right)\right) \ll \exp\left(-\sum_{p\leq x}\frac{r_F(p)}{p}\right).\end{aligned} \tag{4.1}$$

By (iii) of Lemma 4.1, we have

$$\begin{aligned}\sum_{p\leq x}\frac{r_F(p)}{p} &= \sum_{p\leq x}\frac{r_F(p)\log p}{p}\int_p^x \frac{dt}{t(\log t)^2} + \frac{1}{\log x}\sum_{p\leq x}\frac{r_F(p)\log p}{p}\\ &= \int_2^x \left(\sum_{p\leq t}\frac{r_F(p)\log p}{p}\right)\frac{dt}{t(\log t)^2} + O(1)\\ &= \int_2^x \frac{dt}{t\log t} + O(1) = \log\log x + O(1).\end{aligned}$$

On inserting this formula into (4.1), we obtain the lemma. □

Lemma 4.7. *For $x > 0$ and $d \in \mathbb{N}$, we have*

$$\prod_{\substack{p\leq x\\ p\mid d}}\left(1 - \frac{r_F(p)}{p}\right)^{-1} \ll \left(\frac{d}{\varphi(d)}\right)^k,$$

where the implicit constant depends only on $k = \deg F$.

Proof. Note that $r_F(p) \le k$ since the congruence

$$F(x) \equiv 0 \pmod p$$

can be regarded as an algebraic equation in the field $\mathbb{Z}/p\mathbb{Z}$. We thus have

$$\begin{aligned}\prod_{\substack{p<q\\ p\mid d}}\left(1-\frac{r_F(p)}{p}\right)^{-1} &\le \prod_{p\mid d}\left(1-\frac{r_F(p)}{p}\right)^{-1}\\ &= \exp\left(\sum_{p\mid d}\frac{r_F(p)}{p}+O\left(\sum_{p\mid d}\frac{r_F(p)^2}{p^2}\right)\right)\\ &\ll \exp\left(\sum_{p\mid d}\frac{k}{p}\right)\le\left(\exp\left(\sum_{p\mid d}\log\left(1-\frac{1}{p}\right)^{-1}\right)\right)^k=\left(\frac{d}{\varphi(d)}\right)^k.\end{aligned}$$

This completes the proof. □

4.3. **Smooth values of a polynomial.** We now prove an upper bound for the number of smooth values of a polynomial. We first prepare an auxiliary estimate:

Lemma 4.8. *Let $\ell \in \mathbb{N}$, $x, y \ge 2$, $u := \frac{\log x}{\log y}$ and $z := xy^{-3}$. We then have*

$$\sum_{q\le y}\frac{1}{\log q}\sum_{\substack{d>z\\ p_{\max}(d)=q}}\frac{r_{F,\ell}(d)}{d}\ll g_B(u)$$

provided u is sufficiently large in terms of F and ℓ and

$$y \ge \exp((\log x)^{\frac{2}{3}}), \tag{4.2}$$

where the constant B and the implicit constant depend only on F and ℓ.

Proof. Since u is large, x, y, z, u are all large and we may assume $x \ge y^4$ so that $z \ge x^{\frac14}$. Let $g(u) = g_{B_1}(u)$ as defined in (2.1) with sufficiently large constant $B_1 > 0$ depending on F and ℓ. We first decompose the sum as

$$\sum_{q\le y}\frac{1}{\log q}\sum_{\substack{d>z\\ p_{\max}(d)=q}}\frac{r_{F,\ell}(d)}{d}=\sum_{q\le(\log z)^2}+\sum_{(\log z)^2<q\le y}=\sum\nolimits_1+\sum\nolimits_2,\quad\text{say.} \tag{4.3}$$

For the sum $\sum_1$, by noting $q \le (\log z)^2$ and using Lemma 4.2,

$$\begin{aligned}\sum\nolimits_1 &\le \sum_{q\le(\log z)^2}\sum_{\substack{d>z\\ p_{\max}(d)\le\log z}}\frac{r_{F,\ell}(d)}{d}\\ &\ll \pi((\log z)^2)(\log\log z)g\left(\frac{\log z}{\log\log z}\right)\ll(\log z)^2 g\left(\frac{\log z}{\log\log z}\right)\le z^{-\frac12}\end{aligned}$$

for large z. Since $z \ge x^{\frac14}$ and (4.2) implies

$$z^{\frac12}\ge x^{\frac18}\ge\exp((\log x)^{\frac23})\ge\exp(u^2)\ge\exp(u\log u+u\log\log(u+e))\ge\frac{1}{g(u)}$$

for large u, we have

$$\sum\nolimits_1 \ll g(u). \tag{4.4}$$

For the sum $\sum_2$, by noting $q > (\log z)^2 \geq \log z \log\log z$ and by using Lemma 4.3, we have

$$\sum\nolimits_2 \ll \sum_{(\log z)^2 < q \leq y} \frac{1}{q} g\left(\frac{\log z}{\log q}\right). \tag{4.5}$$

By Lemma 3.4, for $q \leq y$, we have

$$\begin{aligned} g\left(\frac{\log z}{\log q}\right) &= g\left(\frac{\log z}{\log y} + \frac{\log z}{\log y}\frac{\log \frac{y}{q}}{\log q}\right) \\ &\leq g\left(\frac{\log z}{\log y}\right)\left(e^{-B}\frac{\log z}{\log y}\log\left(\frac{\log z}{\log y}\right)\right)^{-\frac{\log z}{\log y}\frac{\log \frac{y}{q}}{\log q}}. \end{aligned}$$

Since $\frac{\log z}{\log y} \geq \frac{u}{4}$ by $z \geq x^{\frac{1}{4}}$, we have

$$g\left(\frac{\log z}{\log q}\right) \leq g\left(\frac{\log z}{\log y}\right)e^{-\frac{\log z}{\log y}\frac{\log \frac{y}{q}}{\log q}} \leq g\left(\frac{\log z}{\log y}\right)e^{-\frac{\log \frac{y}{q}}{\log q}} \ll g\left(\frac{\log z}{\log y}\right)e^{-\frac{\log y}{\log q}}$$

for large u. By using this estimate in (4.5) and noting that

$$e^{-\frac{\log y}{\log q}} = \frac{1}{e^{\frac{\log y}{\log q}}} = \frac{1}{1 + (\frac{\log y}{\log q}) + \frac{1}{2}(\frac{\log y}{\log q})^2 + \cdots} \leq \frac{\log y}{\log q},$$

we have

$$\sum\nolimits_2 \ll g\left(\frac{\log z}{\log y}\right)\sum_{q \leq y}\frac{1}{q}e^{-\frac{\log y}{\log q}} \ll g\left(\frac{\log z}{\log y}\right)\frac{1}{\log y}\sum_{q \leq y}\frac{\log q}{q} = g\left(\frac{\log z}{\log y}\right) = g(u-3),$$

where we estimated the sum over q by Mertens' theorem. By using Lemma 3.4 again,

$$\sum\nolimits_2 \ll g(u)u^3(\log(u+e))^3 = g(u)\exp(O(\log u)). \tag{4.6}$$

On inserting (4.4), (4.6) into (4.3), we obtain the assertion. □

Proof of Theorem 1.3. We may assume x, y, u are sufficiently large. In particular, we may assume $x \geq y^4$. Let $g(u) = g_{B_1}(u)$ as defined in (2.1) with sufficiently large constant $B_1 > 0$ depending on F. By (1.4), we then have

$$g(u) \geq \exp\left(-u\log\frac{\log x}{\log\log x} - u\log\log\log x + \frac{1}{2}u\right) = \exp\left(-u\log\log x + \frac{1}{2}u\right) \tag{4.7}$$

for large u. We consider two cases according to the size of y.

We first consider the case

$$y \geq \exp((\log x)^{\frac{2}{3}}). \tag{4.8}$$

Let $z \coloneqq xy^{-3} \geq x^{\frac{1}{4}} \geq y$. Then, (4.8) implies

$$\frac{1}{g(u)} \leq \exp(u\log u + u\log\log(u+e)) \leq \exp(u^2) \leq \exp((\log x)^{\frac{2}{3}}) \leq y$$

so that $z \leq xg(u)$ for large x, u. Thus, it suffices to prove

$$\sum_{\substack{2z < n \leq x \\ p_{\max}(F(n)) \leq y}} 1 \ll xg(u)\exp(O(\log u)). \tag{4.9}$$

For a given integer n satisfying $2z < n \le x$ and $p_{\max}(F(n)) \le y$, write

$$F(n) = p_1 \cdots p_r \quad \text{with} \quad y \ge p_1 \ge \cdots \ge p_r.$$

Since $2z < n \le x$ implies

$$F(n) \ge \left(\frac{n}{2}\right)^k > z^k \ge z \ge y \ge p_r$$

for large x, we may find $i \in \{1, \dots, r-1\}$ such that

$$p_i \cdots p_r > z \ge p_{i+1} \cdots p_r.$$

Then, by writing $q \coloneqq p_i$ and $d \coloneqq p_i \cdots p_r$, we have

$$q \le y, \quad z < d \le zq, \quad p_{\max}(d) = q, \quad d \mid F(n) \quad \text{and} \quad p_{\min}(F(n)/d) \ge q.$$

Therefore, our sum (4.9) is bounded as

$$\sum_{\substack{2z<n\le x\\ p_{\max}(F(n))<y}} 1 \le \sum_{q\le y} \sum_{\substack{z<d\le zq\\ p_{\max}(d)=q}} \sum_{\substack{n\le x\\ d\mid F(n)\\ p_{\min}(F(n)/d)\ge q}} 1 = \sum_{q\le y} \sum_{\substack{z<d\le zq\\ p_{\max}(d)=q}} \sum_{\substack{1\le a\le d\\ F(a)\equiv 0 \ (\mathrm{mod}\ d)}} \Phi(x,q;d,a), \tag{4.10}$$

where

$$\Phi(x,q;d,a) \coloneqq \sum_{\substack{n\le x\\ n\equiv a \ (\mathrm{mod}\ d)\\ p_{\min}(F(n)/d)\ge q}} 1.$$

We estimate $\Phi(x,q;d,a)$. We first rewrite the sum slightly as

$$\Phi(x,q;d,a) \le \sum_{\substack{n\le x/d\\ p_{\min}(F(dn+a)/d)\ge q}} 1 =: S(x,q;d,a), \quad \text{say.} \tag{4.11}$$

For a prime p with $p < q$, let $\Omega(p)$ be sets of residues (mod p) defined by

$$\Omega(p) \coloneqq \begin{cases} \{b \ (\mathrm{mod}\ p) \mid F(db+a) \equiv 0 \ (\mathrm{mod}\ p)\} & \text{if } (d,p) = 1,\\ \varnothing & \text{if } (d,p) > 1.\end{cases}$$

Let $\omega(p) \coloneqq \#\Omega(p)$. Then, the quantity $\Phi(x,q;d,a)$ given in (4.11) can be bounded as

$$S(x,q;d,a) \le \#\{n \le x/d \mid n \ (\mathrm{mod}\ p) \notin \Omega(p) \text{ for all } p < q\} \tag{4.12}$$

We shall apply Lemma 4.4 to estimate the right-hand side and then use Lemma 4.5 to bound the resulting sum $G(y)$. To this end, we check assumptions of these lemmas. Note that since $F(T)$ is a primitive polynomial, for any prime $p \le y$, we have

$$\omega(p) \le r_F(p) \le k, \tag{4.13}$$

where k is the degree of F. If we have $\omega(p) = p$ for some prime $p < q$, then we obviously have $S(x,q;d,a) = 0$ so we may assume $\omega(p) < p$ for all $p < q$ to bound $S(x,q;d,a)$. Then, by (4.13) and Mertens' estimate, we find that the current $\omega(p)$ satisfies the assumptions (Ω_1) and $(\Omega_2(\kappa))$ of Lemma 4.5 with $\kappa = k$ and some constant $A \ge 1$ depending on k. By recalling $z = xy^{-3}$, we have

$$\left(\frac{x}{d}\right)^{\frac12} \ge \left(\frac{x}{zq}\right)^{\frac12} \ge \left(\frac{x}{zy}\right)^{\frac12} = y.$$

Thus, by applying Lemma 4.4 and Lemma 4.5 to (4.12), we obtain

$$S(x,q;d,a) \ll \prod_{\substack{p<q\\ p\nmid d}}\left(1-\frac{\omega(p)}{p}\right)\frac{x}{d}$$

Note that $\omega(p)=r_F(p)$ if $(d,p)=1$. Thus, by using (4.11), Lemma 4.6 and Lemma 4.7,

$$\Phi(x,q;d,a) \ll \prod_{\substack{p<q\\ p\nmid d}}\left(1-\frac{r_F(p)}{p}\right)\frac{x}{d} \ll \frac{1}{d}\left(\frac{d}{\varphi(d)}\right)^k \prod_{p<q}\left(1-\frac{r_F(p)}{p}\right)x \ll \frac{1}{d}\left(\frac{d}{\varphi(d)}\right)^k \frac{x}{\log q}. \tag{4.14}$$

On inserting (4.14) into (4.10) and using Lemma 4.8 with recalling (4.8), we arrive at

$$\sum_{\substack{2z<p\le x\\ p_{\max}(F(p))<y}} 1 \ll x\sum_{q\le y}\frac{1}{\log q}\sum_{\substack{z<d\le zq\\ p_{\max}(d)=q}}\frac{r_F(d)}{d}\left(\frac{d}{\varphi(d)}\right)^k \ll x\sum_{q\le y}\frac{1}{\log q}\sum_{\substack{d>z\\ p_{\max}(d)=q}}\frac{r_{F,k}(d)}{d} \ll xg(u).$$

This completes the proof for the case (4.8).

We next consider the case

$$\log x\log\log x \le y \le \exp((\log x)^{\frac{2}{3}}). \tag{4.15}$$

We first dissect the sum dyadically to obtain

$$\sum_{\substack{n\le x\\ p_{\max}(F(n))\le y}} 1 \ll (\log x)\sup_{xg(u)\le U\le x}\sum_{\substack{U<n\le 2U\\ p_{\max}(F(n))\le y}} 1 + xg(u).$$

Then it suffices to prove

$$\sum_{\substack{U<n\le 2U\\ p_{\max}(F(n))\le y}} 1 \le (x\log y)g(u)\exp(O(\log u)) \tag{4.16}$$

for $xg(u)\le U\le x$ since

$$(\log x)(\log y) \le \exp(2\log\log x) = \exp(O(\log u)).$$

by (4.15). For $U<n\le 2U$, write

$$F(n) = p_1p_2\cdots p_r \quad \text{with} \quad y\ge p_1\ge p_2\ge\cdots\ge p_r$$

and take $i\in\{1,\dots,r\}$ by

$$p_i\cdots p_r > \frac{U}{2} \ge p_{i+1}\cdots p_r =: d, \tag{4.17}$$

which exists since (4.7) implies

$$n>U \implies n>xg(u)\ge x\exp\left(-u\log\log x+\frac{1}{2}u\right)\ge\exp\left(\frac{1}{2}u\right)$$

so that

$$F(n)\ge\frac{n^k}{2}>\frac{U^k}{2}\ge\frac{U}{2} \quad \text{for} \quad U<n\le 2U$$

provided u is large. If $p_{\max}(F(n))\le y$, then (4.17) implies

$$\frac{U}{2}\ge d>\frac{U}{2y}.$$

Also, note that $p_{\max}(D(n)) \le y$. Therefore,

$$\sum_{\substack{U<n\le 2U\\ p_{\max}(F(n))\le y}} 1 \le \sum_{\substack{\frac{U}{2y}<d\le\frac{U}{2}\\ p_{\max}(d)\le y}} \sum_{\substack{U<n\le 2U\\ d\mid F(n)}} 1 \ll U \sum_{\substack{d>\frac{U}{2y}\\ p_{\max}(d)\le y}} \frac{r_F(d)}{d}. \tag{4.18}$$

We apply Lemma 4.2 to the last sum. To this end, we check the assumptions

$$\frac{U}{2y} \ge 1 \quad \text{and} \quad y \ge \log\frac{U}{2y}\log\log\frac{U}{2y}$$

of Lemma 4.2. We start with $\frac{U}{2y} \ge 1$. By (4.15), we have

$$u = \frac{\log x}{\log y} \le \frac{\log x}{\log\log x + \log\log\log x}$$

so that by (4.15) and (4.7), we obtain

$$\frac{U}{2y} \ge \frac{xg(u)}{2y} \ge \frac{x\exp(-u\log\log x)}{2y} \ge \frac{1}{2}\exp\left((\log x)\frac{\log\log\log x}{\log\log x+\log\log\log x} - (\log x)^{\frac{2}{3}}\right) \ge 1$$

for large x, u. The assumption $y \ge \log\frac{U}{2y}\log\log\frac{U}{2y}$ immediately follows by (4.15) since $\frac{U}{2y} \le U \le x$. Thus, we may apply Lemma 4.2 to the sum in (4.18) to obtain

$$\sum_{\substack{U<n\le 2U\\ p_{\max}(F(n))\le y}} 1 \ll U(\log y)g\left(\frac{\log\frac{U}{2y}}{\log y}\right). \tag{4.19}$$

By Lemma 3.4, we have

$$\begin{aligned} g\left(\frac{\log\frac{U}{2y}}{\log y}\right) &= g\left(\frac{\log U}{\log y} - \frac{\log 2y}{\log y}\right)\\ &\le g\left(\frac{\log U}{\log y}\right)\left(e^3\frac{\log U}{\log y}\log\left(\frac{\log U}{\log y}+e\right)\right)^{\frac{\log 2y}{\log y}}\\ &\le g\left(\frac{\log U}{\log y}\right)(e^3u\log(u+e))^{\frac{\log 2y}{\log y}} \le g\left(\frac{\log U}{\log y}\right)\exp(O(\log u)) \end{aligned} \tag{4.20}$$

for large x and B. Again by Lemma 3.4, we further have that

$$g\left(\frac{\log U}{\log y}\right) = g\left(\frac{\log x}{\log y} - \frac{\log\frac{x}{U}}{\log y}\right) \le g(u)(e^3u\log(u+e))^{\frac{\log\frac{x}{U}}{\log y}}. \tag{4.21}$$

By (4.15), we have

$$u\log(u+e) \le \frac{\log x}{\log y}\log\log x \le \log x \le y$$

so that by (4.21), we have

$$g\left(\frac{\log U}{\log y}\right) \le \frac{x}{U}g(u). \tag{4.22}$$

On inserting (4.20) and (4.22) into (4.19), we obtain (4.16). This completes the proof. □

4.4. **Smooth values of a polynomial over primes.** We now prove Theorem 1.4.

Proof of Theorem 1.4. By Theorem 1.3, we may assume that

$$y \geq \exp((\log x)^{\frac{2}{3}}) \tag{4.23}$$

since otherwise we may use $\pi_F(x, y) \leq \Psi_F(x, y)$ and

$$\log x = \exp(\log\log x) = \exp(O(\log u))$$

to reduce the assertion to Theorem 1.3. We may assume that x, y, u are sufficiently large. In particular, we may assume $x \geq y^4$. Let $g(u) = g_{B_1}(u)$ as defined in (2.1) with sufficiently large constant $B_1 > 0$ depending on F. Note that by (4.23), we have

$$\frac{1}{g(u)} = \exp(u \log u + u \log\log(u + e)) \leq \exp(u^2) \leq \exp((\log x)^{\frac{2}{3}}) \leq y \tag{4.24}$$

for large x, u.

We then show that the lemma trivially holds for the case $F(T) = T$. Indeed, for the case $F(T) = T$, we have $\pi_F(x, y) = \pi(y)$. By (4.24) and $x \geq y^4$, we have

$$\pi_F(x, y) = \pi(y) \ll y \ll y^2 g(u) \ll \frac{x}{\log x} g(u).$$

Hence, the lemma is trivial for the case $F(T) = T$ and we may assume $F(T) \neq T$.

Let $z \coloneqq xy^{-3} \geq x^{\frac{1}{4}} \geq y$. Then, since $z \leq \frac{x}{\log x} g(u)$ for large x, u by (4.24), it suffices to prove

$$\sum_{\substack{2z < p \leq x \\ p_{\max}(F(p)) \leq y}} 1 \ll \frac{x}{\log x} g(u) \exp(O(\log u)). \tag{4.25}$$

For a given prime p satisfying $2z < p \leq x$ and $p_{\max}(F(p)) \leq y$, write

$$F(p) = p_1 \cdots p_r \quad \text{with} \quad y \geq p_1 \geq \cdots \geq p_r.$$

Since $2z < p \leq x$ implies

$$F(p) \geq \left(\frac{p}{2}\right)^k > z^k \geq z \geq y \geq p_r$$

for large x, we may find $i \in \{1, \ldots, r-1\}$ such that

$$p_i \cdots p_r > z \geq p_{i+1} \cdots p_r.$$

Then, by writing $q \coloneqq p_i$ and $d \coloneqq p_i \cdots p_r$, we have

$$q \leq y, \quad z < d \leq zq, \quad p_{\max}(d) = q, \quad d \mid F(p) \quad \text{and} \quad p_{\min}(F(p)/d) \geq q.$$

Therefore, our sum (4.25) is bounded as

$$\begin{aligned} \sum_{\substack{2z < p \leq x \\ p_{\max}(F(p)) < y}} 1 &\leq \sum_{q \leq y} \sum_{\substack{z < d \leq zq \\ p_{\max}(d) = q}} \sum_{\substack{p \leq x \\ d \mid F(p) \\ p_{\min}(F(p)/d) \geq q}} 1 \\ &= \sum_{q \leq y} \sum_{\substack{z < d \leq zq \\ p_{\max}(d) = q}} \sum_{\substack{1 \leq a \leq d \\ F(a) \equiv 0 \ (\mathrm{mod}\ d)}} \Phi(x, q; d, a), \end{aligned} \tag{4.26}$$

where

$$\Phi(x,q;d,a) \coloneqq \sum_{\substack{p\le x\\ p\equiv a \,(\mathrm{mod}\ d)\\ p_{\min}(F(p)/d)\ge q}} 1.$$

We estimate $\Phi(x,q;d,a)$. For the case $(a,d)>1$, we have

$$\Phi(x,q;d,a)\le 1. \tag{4.27}$$

For the case $(a,d)=1$, we use a sieve method. We first rewrite the sum slightly as

$$\begin{aligned}\Phi(x,q;d,a) &\le \sum_{\substack{y<dn+a\le x\\ dn+a:\ \text{prime}\\ p_{\min}(F(dn+a)/d)\ge q}} 1+y\\ &\le \sum_{\substack{n\le x/d\\ \forall p\nmid d,\ p\mid dn+a\ \Rightarrow\ p>y\\ \forall p\nmid d,\ p\mid F(dn+a)\ \Rightarrow\ p\ge q}} 1+y = S(x,q;d,a)+y, \quad \text{say.}\end{aligned} \tag{4.28}$$

For a prime p with $p\le y$, let $\Omega(p)$ be sets of residues (mod p) defined by

$$\Omega(p)\coloneqq\{b \ (\mathrm{mod}\ p)\mid db+a\equiv 0 \ (\mathrm{mod}\ p)\}\cup\{b \ (\mathrm{mod}\ p)\mid F(db+a)\equiv 0 \ (\mathrm{mod}\ p)\} \tag{4.29}$$

if $2\le p<q$ and $p\nmid d$

$$\Omega(p)\coloneqq\{b \ (\mathrm{mod}\ p)\mid db+a\equiv 0 \ (\mathrm{mod}\ p)\}.$$

if $q\le p\le y$ and $p\nmid d$, and $\Omega(p)=\varnothing$ if $2\le p\le y$ with $p\mid d$. Let $\omega(p)\coloneqq\#\Omega(p)$. Then, the quantity $S(x,q;d,a)$ given in (4.28) can be written as

$$S(x,q;d,a)=\#\{n\le x/d\mid n \,(\mathrm{mod}\ p)\notin\Omega(p) \text{ for all } p\le y\}. \tag{4.30}$$

We shall apply Lemma 4.4 to estimate the right-hand side and then use Lemma 4.5 to bound the resulting sum $G(y)$. To this end, we check assumptions of these lemmas. Note that since $F(X)$ is a primitive polynomial, for any prime $p\le y$, we have

$$\omega(p)\le 1+r_F(p)\le 1+k, \tag{4.31}$$

where k is the degree of F. If we have $\omega(p)=p$ for some prime $p\le y$, then we obviously have $S(x,q;d,a)=0$ so we may assume $\omega(p)<p$ for all $p\le y$ to bound $S(x,q;d,a)$. Then, by (4.31) and Mertens' estimate, we find that the current $\omega(p)$ satisfies the assumptions (Ω_1) and $(\Omega_2(\kappa))$ of Lemma 4.5 with $\kappa=k+1$ and some constant $A\ge 1$ depending on k. By recalling $z=xy^{-3}$, we also have

$$\left(\frac{x}{d}\right)^{\frac12}\ge\left(\frac{x}{zq}\right)^{\frac12}\ge\left(\frac{x}{zy}\right)^{\frac12}=y.$$

Thus, by applying Lemma 4.4 and Lemma 4.5 to (4.30), we obtain

$$S(x,q;d,a)\ll\prod_{\substack{p\le y\\ p\nmid d}}\left(1-\frac{\omega(p)}{p}\right)\frac{x}{d} \tag{4.32}$$

Since we are assuming $F(T)\ne T$ and $F(T)$ is irreducible and primitive, the constant term of $F(T)$, say c_0 is non-zero. Thus, except $O(1)$ exceptional p's consisting of prime factors of c_0, $F(T)$ (mod p)

is not divisible by $T \pmod p$ so that $F(0) \not\equiv 0 \pmod p$. This implies that except $O(1)$ exceptional p's, the union in (4.29) is a disjoint union. Thus, by using Lemma 4.7, we can bound (4.32) as

$$S(x,q;d,a) \ll \prod_{\substack{p\le y\\ p\nmid d}}\left(1-\frac{1}{p}\right)\prod_{\substack{p<q\\ p\nmid d}}\left(1-\frac{r_F(p)}{p}\right)\frac{x}{d} \ll \frac{1}{\varphi(d)}\left(\frac{d}{\varphi(d)}\right)^k\prod_{p<q}\left(1-\frac{r_F(p)}{p}\right)\frac{x}{\log y}.$$

By using Lemma 4.6 and substituting the resulting estimate into (4.28), we obtain

$$\Phi(x,q;d,a) \ll \frac{1}{\varphi(d)}\left(\frac{d}{\varphi(d)}\right)^k\frac{1}{\log q}\frac{x}{\log y}+y. \tag{4.33}$$

By (4.27), this estimate (4.33) is valid for all a. In the range $q \le y$, $z < d \le zq$ as in (4.26),

$$d \le zq \le zy = xy^{-2} \le \frac{x}{y(\log y)^2}$$

so that

$$y \le \frac{d}{\varphi(d)}\frac{\log y}{\log q}\cdot y \le \frac{1}{\varphi(d)}\frac{1}{\log q}\frac{x}{\log y} \le \frac{1}{\varphi(d)}\left(\frac{d}{\varphi(d)}\right)^k\frac{1}{\log q}\frac{x}{\log y}$$

for large y. Therefore, we can simplify (4.33) to

$$\Phi(x,q;d,a) \ll \frac{1}{\varphi(d)}\left(\frac{d}{\varphi(d)}\right)^k\frac{1}{\log q}\frac{x}{\log y}. \tag{4.34}$$

On inserting (4.34) into (4.26), and using Lemma 4.8 with recalling (4.23), we arrive at

$$\begin{aligned}\sum_{\substack{2z<p\le x\\ p_{\max}(F(p))<y}} 1 &\ll \frac{x}{\log y}\sum_{q\le y}\frac{1}{\log q}\sum_{\substack{z<d\le zq\\ p_{\max}(d)=q}}\frac{r_F(d)}{d}\left(\frac{d}{\varphi(d)}\right)^{k+1}\\ &\ll \frac{x}{\log y}\sum_{q\le y}\frac{1}{\log q}\sum_{\substack{d>z\\ p_{\max}(d)=q}}\frac{r_{F,k+1}(d)}{d}\\ &\ll \frac{x}{\log y}g(u) = \frac{xu}{\log x}g(u) = \frac{x}{\log x}g(u)\exp(O(\log u)).\end{aligned}$$

Thus, we obtain (4.25). This completes the proof. □

5. Smooth values of $\sigma(n^k)$

In this section, we shall prove Theorem 1.2.

We first prepare an analog of Hildebrand's formula [8, formula (2), p. 292]:

Lemma 5.1. *For $x, y > 1$, we have*

$$\Sigma_k^\flat(x,y) \le \frac{1}{\log x}\int_1^x \Sigma_k^\flat(t,y)\frac{dt}{t} + \frac{1}{\log x}\sum_{\substack{p\le x\\ p_{\max}(\Phi_{k+1}(p))\le y}}\Sigma_k^\flat\left(\frac{x}{p},y\right)\log p,$$

where $\Phi_{k+1}(X)$ is the $(k+1)$-th cyclotomic polynomial.

Proof. We have

$$\Sigma_k^\flat(x,y) = \frac{1}{\log x} \sum_{\substack{n\le x\\ n\colon \text{square-free}\\ p_{\max}(\sigma(n^k))\le y}} \left(\log\frac{x}{n}\right) + \frac{1}{\log x} \sum_{\substack{n\le x\\ n\colon \text{square-free}\\ p_{\max}(\sigma(n^k))\le y}} \log n.$$

Then, for the former sum, we have

$$\sum_{\substack{n\le x\\ n\colon \text{square-free}\\ p_{\max}(\sigma(n^k))\le y}} \left(\log\frac{x}{n}\right) = \sum_{\substack{n\le x\\ n\colon \text{square-free}\\ p_{\max}(\sigma(n^k))\le y}} \int_n^x \frac{dt}{t} = \int_1^x \Sigma_k^\flat(t,y)\frac{dt}{t}$$

and for the latter sum, by using

$$n\colon \text{squarefree} \implies \log n = \sum_{pd=n} \log p,$$

we have

$$\begin{aligned}\sum_{\substack{n\le x\\ n\colon \text{square-free}\\ p_{\max}(\sigma(n^k))\le y}} \log n &= \sum_{\substack{pd\le x\\ (p,d)=1\\ d\colon \text{square-free}\\ p_{\max}(\sigma((pd)^k))\le y}} \log p\\ &\le \sum_{\substack{p\le x\\ p_{\max}(\sigma(p^k))\le y}} \log p \sum_{\substack{d\le x/p\\ d\colon \text{square-free}\\ p_{\max}(\sigma(d^k))\le y}} 1 \le \sum_{\substack{p\le x\\ p_{\max}(\sigma(p^k))\le y}} \Sigma_k^\flat\left(\frac{x}{p},y\right)\log p.\end{aligned}$$

Since $\Phi_{k+1}(p) \mid p^k + \cdots + 1 = \sigma(p^k)$, we have

$$\sum_{\substack{n\le x\\ n\colon \text{square-free}\\ p_{\max}(\sigma(n^k))\le y}} \log n \le \sum_{\substack{p\le x\\ p_{\max}(\Phi_{k+1}(p))\le y}} \Sigma_k^\flat\left(\frac{x}{p},y\right)\log p$$

By combining the above estimates, we obtain the lemma. □

Recall the definition of the function $h(u)$ given in (2.2).

Lemma 5.2. *For $u \ge t \ge 0$ with large u, we have*

$$h(u-t) \le h(u)\Big(e^{-\delta+\frac{2}{\log(u+e)}}\big(\log(u+e)\big)\big(\log(\log(u+1)+e)\big)\Big)^t.$$

Proof. Let

$$\ell(u) := -u\log\log(u+e) - u\log\log(\log(u+1)+e) + \delta u$$

so that $h(u) = \exp(\ell(u))$. Since

$$\begin{aligned}\ell'(u) = &-\log\log(u+e) - \log\log(\log(u+1)+e) + \delta\\ &- \frac{u}{(u+e)\log(u+e)} - \frac{u}{(u+1)(\log(u+1)+e)(\log(\log(u+1)+e))},\end{aligned}$$

we have

$$\ell'(u) \ge -\log\Big(e^{-\delta+\frac{2}{\log(u+e)}}\big(\log(u+e)\big)\big(\log(\log(u+1)+e)\big)\Big)$$

for large u. Therefore, by the mean value theorem, we have

$$\begin{aligned}\ell(u-t) &= \ell(u) + \ell'(\xi)(-t)\\ &\le \ell(u) + \Big(-\log\Big(e^{-\delta+\frac{2}{\log(\xi+e)}}\big(\log(\xi+e)\big)\big(\log(\log(\xi+1)+e)\big)\Big)\Big)(-t)\\ &\le \ell(u) + t\log\Big(e^{-\delta+\frac{2}{\log(u+e)}}\big(\log(u+e)\big)\big(\log(\log(u+1)+e)\big)\Big),\end{aligned}$$

where $\xi \in [u-t, u]$ and we used the fact

$$e^{\frac{2}{\log(\xi+e)}}\big(\log(\xi+e)\big) \le \max\left(e^2, e^{\frac{2}{\log(u+e)}}\big(\log(u+e)\big)\right) = e^{\frac{2}{\log(u+e)}}\big(\log(u+e)\big)$$

for large u since

$$\frac{d}{d\xi}\left(\log\log(\xi+e) - \frac{2}{\log(\xi+e)}\right) = \frac{1}{(\xi+e)\log(\xi+e)} - \frac{2}{(\xi+e)(\log(\xi+e))^2}$$

for $\xi \ge 0$. By exponentiating both sides, we obtain the lemma. □

Proof of Theorem 1.2. We may assume $x \ge y$ and that x, y, u are all large since otherwise the assertion is trivial. Let $F(X) \coloneqq \Phi_{k+1}(X)$ be the $(k+1)$-th cyclotomic polynomial. Take a sufficiently large constant $B_1 \ge 3$ so that Theorem 1.4 holds for F with the constant $B \coloneqq B_1$. Then, let $g(v) = g_{B_1}(v)$ as defined in (2.1). Also, take $\delta \in (0,1]$ arbitrarily and let $h(v) = h_\delta(v)$ as defined in (2.2). Any implicit constants in this proof may depend on δ. For $y \ge 2$, $v, w \ge 0$, let us consider

$$K(y,v) \coloneqq \frac{\Sigma_k^\flat(y^v, y)}{y^v h(v)}, \quad \hat{K}(y,w) \coloneqq \sup_{0\le v\le w} K(y,v).$$

Then, it suffices to prove that

$$\hat{K}(y,w) \ll 1 \quad \text{as long as} \quad \log y^w \log\log y^w \le y.$$

Note that $y^w \log\log y^w \le y$ implies

$$\frac{2y}{(\log y)^2} \ge \frac{1}{\log y}\frac{2\log y^w \log\log y^w}{\log\log y^w + \log\log\log y^w} \ge w$$

for large y (for small w, this estimate is trivial).

Let $v_0 = v_0(F,\delta)$ be a large constant. For $y \ge 2$ and $v_0 \le v \le w \le \frac{2y}{(\log y)^2}$, by Lemma 5.1,

$$K(y,v) \le K_1(y,v) + K_2(y,v) + K_3(y,v), \tag{5.1}$$

where

$$\begin{aligned}K_1(y,v) &\coloneqq \frac{1}{h(v)y^v\log y^v}\int_1^{y^v}\Sigma_k^\flat(t,y)\frac{dt}{t},\\ K_2(y,v) &\coloneqq \frac{1}{h(v)y^v\log y^v}\sum_{\substack{y<p\le y^v\\ p_{\max}(F(p))\le y}}\Sigma_k^\flat\left(\frac{y^v}{p}, y\right)\log p,\\ K_3(y,v) &\coloneqq \frac{1}{h(v)y^v\log y^v}\sum_{\substack{p\le y\\ p_{\max}(F(p))\le y}}\Sigma_k^\flat\left(\frac{y^v}{p}, y\right)\log p.\end{aligned}$$

We estimate these three terms separately.

For $K_1(y,v)$, by the definition of $\hat{K}(y,w)$,

$$\begin{aligned} K_1(y,v) &\le \frac{1}{h(v)y^v\log y^v}\int_1^{y^v}\hat{K}\left(y,\frac{\log t}{\log y}\right)h\left(\frac{\log t}{\log y}\right)dt \\ &\le \frac{\hat{K}(y,v)}{h(v)y^v\log y^v}\int_1^{y^v}h\left(\frac{\log t}{\log y}\right)dt = \frac{\hat{K}(y,v)}{h(v)y^v v}\int_0^v h(t)y^t dt = \frac{\hat{K}(y,v)}{h(v)v}\int_0^v h(v-t)y^{-t}dt. \end{aligned}$$

By Lemma 5.2 and $v_0 \le v \le \frac{2y}{(\log y)^2}$,

$$K_1(y,v) \le \frac{\hat{K}(y,v)}{v}\int_0^v\left(\frac{\big(\log(v+e)\big)\big(\log(\log(v+1)+e)\big)}{y}\right)^t dt \le \frac{\hat{K}(y,v)}{v}\int_0^v y^{-\frac{t}{2}}dt \le \frac{2\hat{K}(y,v)}{v\log y}$$

for large y. Thus, we have

$$K_1(y,v) \le \frac{1}{4}\hat{K}(y,v) \tag{5.2}$$

provided y is large.

For $K_2(y,v)$, by the definition of $\hat{K}(y,w)$,

$$\begin{aligned} K_2(y,v) &= \frac{1}{h(v)\log y^v}\sum_{\substack{y<p\le y^v \\ p_{\max}(F(p))\le y}}\hat{K}\left(y,v-\frac{\log p}{\log y}\right)h\left(v-\frac{\log p}{\log y}\right)\frac{\log p}{p} \\ &\le \frac{\hat{K}(y,v)}{h(v)\log y^v}\sum_{\substack{y<p\le y^v \\ p_{\max}(F(p))\le y}}h\left(v-\frac{\log p}{\log y}\right)\frac{\log p}{p}. \end{aligned}$$

By partial summation, we can continue the estimate as

$$\begin{aligned} K_2(y,v) &\le \frac{\hat{K}(y,v)}{h(v)\log y^v}\int_y^{y^v}h\left(v-\frac{\log t}{\log y}\right)\frac{\log t}{t}d\pi_F(t,y) \\ &\le \frac{\hat{K}(y,v)}{h(v)y^v}h(0)\pi_F(y^v,y) \\ &\quad + \frac{\hat{K}(y,v)}{h(v)\log y^v}\int_y^{y^v}h'\left(v-\frac{\log t}{\log y}\right)\frac{\log t}{t}\pi_F(t,y)\frac{dt}{t\log y} \\ &\quad\quad + \frac{\hat{K}(y,v)}{h(v)\log y^v}\int_y^{y^v}h\left(v-\frac{\log t}{\log y}\right)\frac{\log t-1}{t^2}\pi_F(t,y)dt. \end{aligned} \tag{5.3}$$

For $v \ge 0$, we have

$$\begin{aligned} h'(v) = \bigg(&-\log\log(v+e)-\log\log(\log(v+1)+e)+\delta-\frac{v}{(v+e)\log(v+e)} \\ &-\frac{v}{(v+1)(\log(v+1)+e)\log(\log(v+1)+e)}\bigg)h(v) \le h(v). \end{aligned} \tag{5.4}$$

By substituting (5.4) into (5.3), we obtain

$$K_2(y,v) \ll \frac{\hat{K}(y,v)}{h(v)y^v}\pi_F(y^v,y) + \frac{\hat{K}(y,v)}{h(v)\log y^v}\int_y^{y^v}h\left(v-\frac{\log t}{\log y}\right)\frac{\log t}{t^2}\pi_F(t,y)dt.$$

Since we have

$$\log t\log\log t \le \log y^v\log\log y^v \le \log y^w\log\log y^w \le y$$

in the above integral, we can use Theorem 1.4 to obtain

$$
\begin{aligned}
K_2(y,v) &\ll \frac{\hat{K}(y,v)}{\log y^v}\frac{g(v)}{h(v)} + \frac{\hat{K}(y,v)}{h(v)\log y^v}\int_y^{y^v} h\left(v - \frac{\log t}{\log y}\right) g\left(\frac{\log t}{\log y}\right)\frac{dt}{t} \\
&= \frac{\hat{K}(y,v)}{\log y^v}\frac{g(v)}{h(v)} + \frac{\hat{K}(y,v)}{h(v)v}\int_1^v h(v-t)g(t)dt.
\end{aligned} \tag{5.5}
$$

For sufficiently large $v \geq v_0$, we have

$$
\frac{g(v)}{h(v)} \leq \left(\frac{e^{1-\delta}\big(\log(v+e)\big)\big(\log(\log(v+1)+e)\big)}{v\log(v+e)}\right)^v \leq 1. \tag{5.6}
$$

Also, for sufficiently large $v \geq v_0$, by Lemma 5.2, we have

$$
\begin{aligned}
h(v-t)g(t) &\leq h(v)\left(\frac{e^{1-\frac{\delta}{2}}\big(\log(v+e)\big)\big(\log(\log(v+1)+e)\big)}{t\log(t+e)}\right)^t \\
&\leq h(v)e^{-\frac{\delta}{4}t}\left(\frac{e^{1-\frac{\delta}{5}}\log v\log\log v}{t\log(t+e)}\right)^t.
\end{aligned} \tag{5.7}
$$

We shall calculate

$$
\sup_{t\geq 0}\left(\frac{e^{1-\frac{\delta}{5}}\log v\log\log v}{t\log(t+e)}\right)^t. \tag{5.8}
$$

Let $\mathscr{L} := e^{1-\frac{\delta}{5}}\log v\log\log v$ and

$$
\phi(t) := -t\log t - t\log\log(t+e) + t\log\mathscr{L} \quad \text{so} \quad \left(\frac{e^{1-\frac{\delta}{5}}\log v\log\log v}{t\log(t+e)}\right)^t = \exp(\phi(t)).
$$

Then, the first and second derivatives of $\phi(t)$ are

$$
\begin{aligned}
\phi'(t) &= -\log(et\log(t+e)) + \log\mathscr{L} - \frac{t}{(t+e)\log(t+e)}, \\
\phi''(t) &= -\frac{1}{t} - \frac{1}{(t+e)\log(t+e)} + \frac{1}{(t+e)(\log(t+e))^2} \\
&\qquad - \frac{e}{(t+e)^2\log(t+e)} - \frac{e}{(t+e)^2(\log(t+e))^2} \leq 0.
\end{aligned} \tag{5.9}
$$

Thus, the supremum (5.8) is indeed the maximum attained at $t = t_0$ determined by $\phi'(t_0) = 0$. By using (5.9), we find that

$$
\log(et_0\log(t_0+e)) = \log\mathscr{L} + O\left(\frac{1}{\log(t_0+e)}\right) = \log\mathscr{L} + O\left(\frac{1}{\log\mathscr{L}}\right) \tag{5.10}
$$

and

$$
\log t_0 = \log\mathscr{L} + O(\log\log(t_0+e)) = \log\mathscr{L} + O(\log\log\mathscr{L})
$$

so that

$$
\log\log(t_0+e) = \log\log\mathscr{L} + O\left(\frac{\log\log\mathscr{L}}{\log\mathscr{L}}\right).
$$

By substituting this again into (5.10), we obtain

$$
\log t_0 = \log\mathscr{L} - \log\log\mathscr{L} - 1 + O\left(\frac{\log\log\mathscr{L}}{\log\mathscr{L}}\right)
$$

so that

$$t_0 = \frac{\mathscr{L}}{e \log \mathscr{L}}\left(1 + O\left(\frac{\log\log \mathscr{L}}{\log \mathscr{L}}\right)\right) \le e^{-\frac{\delta}{6}} \log v$$

for large $v \ge v_0$. By using this estimate and noting that (5.10) and $e^{\frac{\delta}{7}} > 1$ implies

$$\log(et_0 \log(t_0 + e)) \ge \log \mathscr{L} - e^{\frac{\delta}{7}},$$

for large $v \ge v_0$, we find that

$$\sup_{t\ge 0}\left(\frac{e^{1-\frac{\delta}{5}} \log v \log\log v}{t \log(t+e)}\right)^t = \left(\frac{\mathscr{L}}{t_0 \log(t_0+e)}\right)^{t_0} \le \exp(e^{\frac{\delta}{7}} t_0) \le v^{e^{-\frac{\delta}{42}}}.$$

Therefore, by (5.7), we have

$$h(v-t)g(t) \le h(v) v^{e^{-\frac{\delta}{42}}} e^{-\frac{\delta}{4}t}. \tag{5.11}$$

On inserting (5.6) and (5.11) into (5.5), we have

$$K_2(y,v) \ll \frac{\hat{K}(y,v)}{\log y^v} + \frac{\hat{K}(y,v)}{v^{1-e^{-\frac{\delta}{42}}}}$$

with $1 - e^{-\frac{\delta}{42}} > 0$. Thus, we have

$$K_2(y,v) \le \frac{1}{4}\hat{K}(y,v) \tag{5.12}$$

provided $v \ge v_0$.

For $K_3(y,v)$, by the definition of $\hat{K}(y,w)$, $v \ge v_0$, Lemma 5.2 and Mertens' theorem,

$$\begin{aligned} K_3(y,v) &\le \frac{1}{h(v)\log y^v} \sum_{p\le y} \hat{K}\left(y, v - \frac{\log p}{\log y}\right) h\left(v - \frac{\log p}{\log y}\right)\frac{\log p}{p} \\ &\le \frac{\hat{K}(y,v)}{h(v)\log y^v} \sum_{p\le y} h\left(v - \frac{\log p}{\log y}\right)\frac{\log p}{p} \\ &\ll \frac{\big(\log(v+e)\big)\big(\log(\log(v+1)+e)\big)}{\log y^v} \hat{K}(y,v) \sum_{p\le y}\frac{\log p}{p} \ll \frac{(\log v)^2}{v}\hat{K}(y,v) \end{aligned}$$

Therefore, we obtain

$$K_3(y,v) \le \frac{1}{4}\hat{K}(y,v) \tag{5.13}$$

for sufficiently large $v \ge v_0$.

By combining (5.1), (5.2), (5.12) and (5.13), we obtain

$$K(y,v) \le \frac{3}{4}\hat{K}(y,v)$$

for sufficiently large $v \ge v_0$. By taking the supremum over v,

$$\begin{aligned} \hat{K}(y,w) = \sup_{0\le v\le w} K(y,v) &= \max\left(\sup_{0\le v\le v_0} K(y,v), \sup_{v_0<v\le w} K(y,v)\right) \\ &\le \max\left(\sup_{0\le v\le v_0} K(y,v), \frac{3}{4}\sup_{v_0<v\le w} \hat{K}(y,v)\right) \end{aligned}$$

$$\leq \max\left(\hat{K}(y, v_0), \frac{3}{4}\hat{K}(y, w)\right).$$

Since $\hat{K}(y, w) \leq \frac{3}{4}\hat{K}(y, w)$ implies $\hat{K}(y, w) = 0$, we arrive at

$$\hat{K}(y, w) \leq \hat{K}(y, v_0) \leq \frac{1}{h(v_0)} \ll 1.$$

This completes the proof. □

Remark 5.1. Although it is possible to extend the admissible range of Theorem 1.2 to $y \geq \log x$ by weakening the error term estimate $o(u)$ to $O(u)$ (see Remark 3.1), it seems rather difficult to achieve the admissible range

$$y \geq (\log\log x)^{1+\varepsilon} \tag{5.14}$$

as in [2]. In the paper [2], this range corresponds to Proposition 2.3 [2, p. 1374]. To obtain the range (5.14), Banks–Friedlander–Pomerance–Shparlinski used a trick in the formula (2.7) on p. 1376 or in the last paragraph of p. 1377 of [2]. However, their trick seems not available for $\Sigma_k^\flat(x, y)$ since our problem include polynomials of higher degree.

6. Even-odd amicable pairs

In this section, we prove Theorem 1.1. We start with a simple observation:

Lemma 6.1. *For any even-odd amicable pair (A, B) consisting of an even integer A and an odd integer B, there are positive integers a, M, N such that*

$$A = 2^a M^2, \quad B = N^2, \quad M, N\colon \text{odd}.$$

Proof. Since (A, B) is an amicable pair, we have

$$\sigma(A) = \sigma(B) = A + B. \tag{6.1}$$

By the assumption that A is even and B is odd, we find that $A + B$ is odd. By (6.1), we find that both of $\sigma(A)$ and $\sigma(B)$ are odd. Suppose that p is an odd prime and $p^e \parallel A$. Then, $\sigma(p^e) \mid \sigma(A)$ so $\sigma(p^e)$ should be odd. Since p is odd, this implies

$$1 \equiv \sigma(p^e) \equiv 1 + p + \cdots + p^e \equiv e + 1 \pmod 2,$$

i.e. e is even. This shows that $A = 2^a M^2$ for some positive integer a and odd number M. We can deal with B similarly. This completes the proof. □

Proof of Theorem 1.1. Let x be a large real number, $a \in \mathbb{N}$ and consider

$$\mathscr{B} = \mathscr{B}(x, a) := \{(M, N) \mid \max(2^a M^2, N^2) \leq x,\ (2^a M^2, N^2)\colon \text{amicable},\ M, N\colon \text{odd}\}. \tag{6.2}$$

We discard several parts of $\mathscr{B}$ to introduce useful restrictions to the variables M and N and estimate the size of the discarded parts.

We first want to introduce the restriction

$$\min(M, N) > x^{\frac{1}{2}} L^{-1} \tag{C1}$$

with some real number $L \geq 4$ chosen later. To this end, we decompose $\mathscr{B}$ as

$$\mathscr{B} = \mathscr{B}^{(1)} \sqcup \mathscr{E}^{(1)},$$

where

$$\mathscr{B}^{(1)} := \{(M,N)\in\mathscr{B} \mid \min(M,N) > x^{\frac{1}{2}}L^{-1}\},$$
$$\mathscr{E}^{(1)} := \{(M,N)\in\mathscr{B} \mid \min(M,N) \le x^{\frac{1}{2}}L^{-1}\}.$$

In the subsequent argument, we typically denote the part of $\mathscr{B}$ remaining after the i-th step of our discarding process by $\mathscr{B}^{(i)}$ and the part of $\mathscr{B}$ discarded at the i-th step by $\mathscr{E}^{(i)}$. We refer to the condition used to define $\mathscr{B}^{(i)}$ by the condition "Ci". Also, we refer to each lemma corresponding to the i-th step of discarding process by "**Claim 6.**i".

Claim 6.1. *We have*

$$\#\mathscr{E}^{(1)} \ll x^{\frac{1}{2}}L^{-1},$$

where the implicit constant is absolute.

Proof. It suffices to bound the size of sets

$$\{(M,N)\in\mathscr{B} \mid M \le x^{\frac{1}{2}}L^{-1}\} \quad \text{and} \quad \{(M,N)\in\mathscr{B} \mid N \le x^{\frac{1}{2}}L^{-1}\}$$

since these sets cover $\mathscr{E}^{(1)}$. When a is fixed, for $(M,N)\in\mathscr{B}$, the value of M is determined uniquely by N and the value of N is determined uniquely by M since $(2^aM^2, N^2)$ forms an amicable pair. Thus, the last quantity is bounded by

$$2\cdot\#\{M\in\mathbb{N} \mid M \le x^{\frac{1}{2}}L^{-1}\} \le 2x^{\frac{1}{2}}L^{-1}.$$

This completes the proof. □

For a positive integer m and a real number $z \ge 1$, we let

$$D_z(m) := \prod_{\substack{p^v \| m \\ p \le z}} p^v,$$

which is "the z-smooth part" of m. We next introduce the restriction

$$\text{(C2)} \qquad \max(D_{L^4}(M), D_{L^4}(N)) \le x^{\alpha}$$

with some $\alpha \in (0, \frac{1}{2})$ chosen later. We decompose $\mathscr{B}^{(1)}$ as

$$\mathscr{B}^{(1)} = \mathscr{B}^{(2)} \sqcup \mathscr{E}^{(2)},$$

where

$$\mathscr{B}^{(2)} := \{(M,N)\in\mathscr{B}^{(1)} \mid \max(D_{L^4}(M), D_{L^4}(N)) \le x^{\alpha}\},$$
$$\mathscr{E}^{(2)} := \{(M,N)\in\mathscr{B}^{(1)} \mid \max(D_{L^4}(M), D_{L^4}(N)) > x^{\alpha}\}.$$

In what follows, we let

$$u := \frac{\log x}{\log L}.$$

Claim 6.2. *We have*

$$\#\mathscr{E}^{(2)} \ll x^{\frac{1}{2}}\log x \exp\bigl(-\tfrac{\alpha}{4}u\log u\bigr)$$

provided

$$(\mathbf{L}) \qquad L \ge \log x$$

where the implicit constant is absolute.

Proof. Similarly to the proof of Claim 6.1, it suffices to bound

$$\#\{M \in \mathbb{N} \mid M \le x^{\frac{1}{2}} \text{ and } D_{L^4}(M) > x^{\alpha}\}.$$

By writing $d \coloneqq D_{L^4}(M)$ and $M = dm$, this is further bounded by

$$\le \sum_{\substack{dm \le x^{\frac{1}{2}} \\ p_{\max}(d) \le L^4 \\ d > x^{\alpha}}} 1 = \sum_{\substack{x^{\alpha} < d \le x^{\frac{1}{2}} \\ p_{\max}(d) \le L^4}} \sum_{m \le x^{\frac{1}{2}}/d} 1 \le x^{\frac{1}{2}} \sum_{\substack{x^{\alpha} < d \le x^{\frac{1}{2}} \\ p_{\max}(d) \le L^4}} \frac{1}{d} \le x^{\frac{1}{2}} \sum_{\substack{d > x^{\alpha} \\ p_{\max}(d) \le L^4}} \frac{1}{d}$$

We then apply Lemma 3.3 with the constant function $f(n) = 1$. The assumption (**L**) assures $L^4 \ge \log x^{\alpha} \log\log x^{\alpha}$. Thus, we can apply Lemma 3.3 to obtain the assertion since

$$\frac{\log x^{\alpha}}{\log L^4} = \frac{\alpha}{4}\frac{\log x}{\log L} = \frac{\alpha}{4}u.$$

This completes the proof. □

In the subsequent argument, we always assume (**L**). For a positive integer m, let

$$D^{\sharp}(m) \coloneqq \prod_{\substack{p^v \| m \\ v \ge 2}} p^v,$$

which is "the square-full part" of m. Our next restriction is

$$\text{(C3)} \qquad \max(D^{\sharp}(M), D^{\sharp}(N)) \le L^2$$

and so let

$$\mathscr{B}^{(2)} = \mathscr{B}^{(3)} \sqcup \mathscr{E}^{(3)},$$

where

$$\mathscr{B}^{(3)} \coloneqq \{(M,N) \in \mathscr{B}^{(2)} \mid \max(D^{\sharp}(M), D^{\sharp}(N)) \le L^2\},$$
$$\mathscr{E}^{(3)} \coloneqq \{(M,N) \in \mathscr{B}^{(2)} \mid \max(D^{\sharp}(M), D^{\sharp}(N)) > L^2\}.$$

Claim 6.3. *We have*

$$\#\mathscr{E}^{(3)} \ll x^{\frac{1}{2}} L^{-1},$$

where the implicit constant is absolute.

Proof. As before, it suffices to bound

$$\#\{M \in \mathbb{N} \mid M \le x^{\frac{1}{2}},\ D^{\sharp}(M) > L^2\}.$$

By writing $d \coloneqq D^{\sharp}(M)$ and $M = dm$, this is bounded by

$$\le \sum_{\substack{dm \le x^{\frac{1}{2}} \\ d\colon \text{square-full} \\ d > L^2}} 1 \le x^{\frac{1}{2}} \sum_{\substack{L^2 < d \le x^{\frac{1}{2}} \\ d\colon \text{square-full}}} \frac{1}{d}. \tag{6.3}$$

Since every square-full number d can be written as $d = e^2 f^3$ with some positive integers e and f, we can obtain the bound

$$\sum_{\substack{d > U \\ d\colon \text{square-full}}} \frac{1}{d} \le \sum_{f=1}^{\infty} \frac{1}{f^3} \sum_{e > (U/f^3)^{\frac{1}{2}}} \frac{1}{e^2} \ll \frac{1}{U^{\frac{1}{2}}} \sum_{f=1}^{\infty} \frac{1}{f^{\frac{3}{2}}} \ll \frac{1}{U^{\frac{1}{2}}}.$$

By using this bound in (6.3), we obtain the claim. □

We now give an observation on the remaining part $\mathscr{B}^{(3)}$.

Lemma 6.2. *For any $(M,N)\in\mathscr{B}^{(3)}$, we have*

$$L^4 < p_{\max}(M) \parallel M \quad \text{and} \quad L^4 < p_{\max}(N) \parallel N$$

provided

$$x^{\alpha}, x^{\frac{1}{2}-\alpha} > L^2. \tag{\textsf{LA}}$$

Proof. We have $x^{\alpha} < x^{\frac{1}{2}}L^{-1}$ by (**LA**). Then, by (C1) and (C2), we find that

$$M/D_{L^4}(M), N/D_{L^4}(N) > (x^{\frac{1}{2}}L^{-1})/x^{\alpha} > 1$$

so that $p_{\max}(M), p_{\max}(N) > L^4$. If $p_{\max}(M)^2 \mid M$ or $p_{\max}(N)^2 \mid N$, this implies

$$L^8 < p_{\max}(M)^2 \le D^{\sharp}(M) \quad \text{or} \quad L^8 < p_{\max}(N)^2 \le D^{\sharp}(N),$$

respectively, which contradicts (C3). This completes the proof. $\square$

From now on, we further assume (**LA**). Then, by Lemma 6.2, we may write $(M,N)\in\mathscr{B}^{(3)}$ as

$$\begin{gathered} M = pm, \quad N = qn, \quad p \coloneqq p_{\max}(M), \quad q \coloneqq p_{\max}(N), \\ \text{with conditions} \quad \max(p_{\max}(m), L^4) < p \quad \text{and} \quad \max(p_{\max}(n), L^4) < q. \end{gathered} \tag{6.4}$$

Note that this factorization is clearly unique and $(p,m)=(q,n)=1$.

We next introduce the restriction

$$p_{\max}(M,N) \le L^2, \tag{C4}$$

where, hereafter, we use an abbreviation $p_{\max}(M,N) \coloneqq p_{\max}((M,N))$. Let

$$\mathscr{B}^{(3)} = \mathscr{B}^{(4)} \sqcup \mathscr{E}^{(4)},$$

where

$$\mathscr{B}^{(4)} \coloneqq \{(M,N)\in\mathscr{B}^{(3)} \mid p_{\max}(M,N) \le L^2\},$$
$$\mathscr{E}^{(4)} \coloneqq \{(M,N)\in\mathscr{B}^{(3)} \mid p_{\max}(M,N) > L^2\}.$$

To bound $\#\mathscr{E}^{(4)}$, we need the next lemma.

Lemma 6.3. *For $x\ge 1$ and any positive integer k and any prime number P, we have*

$$\sum_{\substack{\varpi^e \le x \\ P\mid\sigma(\varpi^{ke})}} \frac{1}{\varpi^e} \ll \frac{(\log x)^2}{P^{\frac{1}{k}}}.$$

where ϖ and e runs through prime numbers and positive integers, respectively, and the implicit constant depends only on k.

Proof. We first classify the summands according to the value of e as

$$\sum_{\substack{\varpi^e \le x \\ P\mid\sigma(\varpi^{ke})}} \frac{1}{\varpi^e} = \sum_{e=1}^{O(\log x)} \sum_{\substack{\varpi \le x^{\frac{1}{e}} \\ P\mid\sigma(\varpi^{ke})}} \frac{1}{\varpi^e} \tag{6.5}$$

and then we estimate the inner sums. We first observe that

$$P \mid \sigma(\varpi^{ke}) \implies P \le \sigma(\varpi^{ke}) < \varpi^{ke}\left(1+\frac{1}{\varpi}+\cdots\right) \le 2\varpi^{ke}.$$

Thus, we can restrict the range of ϖ as

$$\sum_{\substack{\varpi\le x^{\frac{1}{e}}\\ P\mid\sigma(\varpi^{ke})}}\frac{1}{\varpi^e}=\sum_{\substack{(P/2)^{\frac{1}{ke}}<\varpi\le x^{\frac{1}{e}}\\ P\mid\sigma(\varpi^{ke})}}\frac{1}{\varpi^e}.$$

By dissecting the sum dyadically,

$$\begin{aligned}\sum_{\substack{\varpi\le x^{\frac{1}{e}}\\ P\mid\sigma(\varpi^{ke})}}\frac{1}{\varpi^e}&\ll(\log x^{\frac{1}{e}})\sup_{(P/2)^{\frac{1}{ke}}\le U\le x^{\frac{1}{e}}}\sum_{\substack{U<\varpi\le 2U\\ P\mid\sigma(\varpi^{ke})}}\frac{1}{\varpi^e}\\&\ll(\log x^{\frac{1}{e}})\sup_{(P/2)^{\frac{1}{ke}}\le U\le x^{\frac{1}{e}}}U^{-e}\sum_{\substack{U<\varpi\le 2U\\ \omega^{ke}+\cdots+1\equiv 0\,(\mathrm{mod}\ P)}}1.\end{aligned}$$

The congruence $\omega^{ke}+\cdots+1\equiv 0\,(\mathrm{mod}\ P)$ is an algebraic equation of degree ke in the finite field of order P and so has only ke solutions at most. Therefore,

$$\begin{aligned}\sum_{\substack{\varpi\le x^{\frac{1}{e}}\\ P\mid\sigma(\varpi^{ke})}}\frac{1}{\varpi^e}&\ll\frac{1}{e}(\log x)\sup_{(P/2)^{\frac{1}{ke}}\le U\le x^{\frac{1}{e}}}eU^{-e}\left(\frac{U}{P}+1\right)\\&=(\log x)\sup_{(P/2)^{\frac{1}{ke}}\le U\le x^{\frac{1}{e}}}\left(\frac{U^{1-e}}{P}+U^{-e}\right).\end{aligned}$$

Since the exponents of U on the right-hand side are non-positive, the supremum is achieved at $U=(P/2)^{\frac{1}{ke}}$. Therefore,

$$\sum_{\substack{\varpi\le x^{\frac{1}{e}}\\ P\mid\sigma(\varpi^{ke})}}\frac{1}{\varpi^e}\ll(\log x)((P^{\frac{1}{ke}}/P)P^{-\frac{1}{k}}+P^{-\frac{1}{k}})\ll\frac{\log x}{P^{\frac{1}{k}}}.$$

On inserting this estimate into (6.5), we obtain the lemma. □

Claim 6.4. *We have*

$$\#\mathscr{E}^{(4)}\ll x^{\frac{1}{2}}(\log x)^2L^{-1},$$

where the implicit constant is absolute.

Proof. For $(M,N)\in\mathscr{B}$, since $(2^aM^2,N^2)$ form an amicable pair, we have

$$(N^2,\sigma(N^2))=(N^2,s(N^2))=(N^2,2^aM^2)$$

(these are equation between the greatest common divisors) so that

$$p_{\max}(M,N)>L^2\implies p_{\max}(N^2,\sigma(N^2))>L^2.$$

Thus, as before, it suffices to bound

$$\#\{N\in\mathbb{N}\mid N\le x^{\frac{1}{2}},\ p_{\max}(N^2,\sigma(N^2))>L^2\}.$$

On writing $p_{\max}(N^2,\sigma(N^2))=P$, this quantity can be bounded as

$$=\sum_{L^2<P\le x^{\frac{1}{2}}}\sum_{\substack{N\le X^{\frac{1}{2}}\\ p_{\max}(N^2,\sigma(N^2))=P}}1\le\sum_{L^2<P\le x^{\frac{1}{2}}}\sum_{\substack{N\le X^{\frac{1}{2}}\\ P\mid(N^2,\sigma(N^2))}}1.$$

For a positive integer N, the condition $P \mid \sigma(N^2)$ implies the existence of some prime power ϖ^e such that $\varpi^e \parallel N$ and $P \mid \sigma(\varpi^{2e})$. In this case, $\sigma(\varpi^{2e})$ is coprime to ϖ so $(\varpi^e, P) = 1$, which tells us $\varpi^e P \mid N$. Therefore, the last quantity is further bounded by using Lemma 6.3 with $k = 2$ as

$$\leq \sum_{L^2 < P \leq x^{\frac{1}{2}}} \sum_{\substack{\varpi^e \leq x^{\frac{1}{2}}/P \\ P \mid \sigma(\varpi^{2e})}} \sum_{\substack{N \leq x^{\frac{1}{2}} \\ \varpi^e P \mid N}} 1 \leq x^{\frac{1}{2}} \sum_{L^2 < P \leq x^{\frac{1}{2}}} \frac{1}{P} \sum_{\substack{\varpi^e \leq x^{\frac{1}{2}} \\ P \mid \sigma(\varpi^{2e})}} \frac{1}{\varpi^e}$$
$$\ll x^{\frac{1}{2}} (\log x)^2 \sum_{L^2 < P \leq x^{\frac{1}{2}}} \frac{1}{P^{\frac{3}{2}}} \ll x^{\frac{1}{2}} (\log x)^2 L^{-1}.$$

This completes the proof. ∎

We then introduce the restriction

$$\text{(C5)} \qquad mn > x^{\frac{1}{2}} L^{-1},$$

where we used the factorization given in (6.4). Let

$$\mathscr{B}^{(4)} = \mathscr{B}^{(5)} \sqcup \mathscr{E}^{(5)},$$

where

$$\mathscr{B}^{(5)} := \{(M, N) \in \mathscr{B}^{(4)} \mid mn > x^{\frac{1}{2}} L^{-1}\},$$
$$\mathscr{E}^{(5)} := \{(M, N) \in \mathscr{B}^{(4)} \mid mn \leq x^{\frac{1}{2}} L^{-1}\}.$$

The estimate for $\#\mathscr{E}^{(5)}$ is based on the following lemma.

Lemma 6.4. *For $a \in \mathbb{N}$ and odd integers m and n, there are at most four odd prime pairs (p, q) such that the pair $(2^a p^2 m^2, q^2 n^2)$ is amicable and $(p, m) = (q, n) = 1$.*

Proof. Assume that an odd prime pair (p, q) is given so that the pair $(2^a p^2 m^2, q^2 n^2)$ is amicable and $(p, m) = (q, n) = 1$. Since the pair $(2^a p^2 m^2, q^2 n^2)$ is amicable, we obtain

$$\sigma(2^a p^2 m^2) = \sigma(q^2 n^2) \quad \text{and} \quad \sigma(2^a p^2 m^2) - 2^a p^2 m^2 = q^2 n^2.$$

By writing $\mu := 2^a m^2$ and $\nu := n^2$, we may rewrite these equations to

$$\sigma(p^2 \mu) = \sigma(q^2 \nu) \quad \text{and} \quad \sigma(p^2 \mu) - p^2 \mu = q^2 \nu.$$

By using $\sigma(p^2) = p^2 + p + 1$, we can further rewrite them to

$$p^2 \sigma(\mu) + p\sigma(\mu) + \sigma(\mu) = q^2 \sigma(\nu) + q\sigma(\nu) + \sigma(\nu) \tag{6.6}$$
$$p^2 s(\mu) + p\sigma(\mu) + \sigma(\mu) = q^2 \nu. \tag{6.7}$$

By multiplying (6.6) and (6.7) by ν and $\sigma(\nu)$, respectively, and taking the difference,

$$p^2(\sigma(\mu)\nu - s(\mu)\sigma(\nu)) - p\sigma(\mu)s(\nu) - \sigma(\mu)s(\nu) = q\sigma(\nu)\nu + \sigma(\nu)\nu,$$

or, equivalently,

$$p^2(\sigma(\mu)\nu - s(\mu)\sigma(\nu)) - p\sigma(\mu)s(\nu) - (\sigma(\mu)s(\nu) + \sigma(\nu)\nu) = q\sigma(\nu)\nu.$$

By taking the square,

$$(p^2(\sigma(\mu)\nu - s(\mu)\sigma(\nu)) - p\sigma(\mu)s(\nu) - (\sigma(\mu)s(\nu) + \sigma(\nu)\nu))^2 = q^2\nu \cdot \sigma(\nu)^2 \nu.$$

By substituting (6.7) again here,

$$(p^2(\sigma(\mu)\nu - s(\mu)\sigma(\nu)) - p\sigma(\mu)s(\nu) - (\sigma(\mu)s(\nu) + \sigma(\nu)\nu))^2 - (p^2 s(\mu) + p\sigma(\mu) + \sigma(\mu))\sigma(\nu)^2\nu = 0. \tag{6.8}$$

If we regard this equation as the quartic equation of the indeterminate p, then the linear term coefficient on the left-hand side is

$$2\sigma(\mu)s(\nu)(\sigma(\mu)s(\nu) + \sigma(\nu)\nu) - \sigma(\mu)\sigma(\nu)^2\nu. \tag{6.9}$$

This is odd since $\nu = n^2$ is odd and $\sigma(\nu), \sigma(\mu)$ are odd since $\sigma(\nu) = \sigma(n^2)$ and $\sigma(\mu) = \sigma(2^a m^2)$. In particular, the coefficient (6.9) is non-zero. Thus, the equation (6.8) is not trivial. This implies that for a given (a, m, n), there are at most four possible values for p. For each of those p, the value of q is uniquely determined by (6.7). This completes the proof. □

Claim 6.5. *We have*

$$\#\mathscr{E}^{(5)} \ll x^{\frac{1}{2}}(\log x)L^{-1},$$

where the implicit constant is absolute.

Proof. By Lemma 6.4, we have

$$\#\mathscr{E}^{(5)} \le 4 \cdot \#\{(m,n) \in \mathbb{N}^2 \mid mn \le x^{\frac{1}{2}}L^{-1}\},$$

which is bounded by

$$\ll \sum_{mn \le x^{\frac{1}{2}}/L} 1 \le \sum_{m \le x^{\frac{1}{2}}/L} \sum_{n \le x^{\frac{1}{2}}/mL} 1 \le x^{\frac{1}{2}}L^{-1} \sum_{m \le x^{\frac{1}{2}}/L} \frac{1}{m} \ll x^{\frac{1}{2}}(\log x)L^{-1}.$$

This completes the proof. □

We further introduce the restriction

$$\max(p,q) \le x^{\frac{1}{2}-\beta} \tag{C6}$$

with some $\beta \in (0, \frac{1}{2})$ chosen later, where we used the factorization given in (6.4). Let

$$\mathscr{B}^{(5)} = \mathscr{B}^{(6)} \sqcup \mathscr{E}^{(6)},$$

where

$$\mathscr{B}^{(6)} := \{(M,N) \in \mathscr{B}^{(5)} \mid \max(p,q) \le x^{\frac{1}{2}-\beta}\},$$
$$\mathscr{E}^{(6)} := \{(M,N) \in \mathscr{B}^{(5)} \mid \max(p,q) > x^{\frac{1}{2}-\beta}\}.$$

We require the next bound for the number of the solutions of polynomial congruence.

Lemma 6.5. *For a polynomial with integral coefficients*

$$a_k x^k + \cdots + a_1 x + a_0 \in \mathbb{Z}[X],$$

a positive integer D, the congruence

$$a_k x^k + \cdots + a_1 x + a_0 \equiv 0 \pmod{D}$$

has at most

$$(a_k, \ldots, a_1, D)^{\frac{1}{k}} D^{1-\frac{1}{k}} \tau(D)^{k-1}$$

solutions (mod D)*.*

Proof. This is a result of Kamke [12, Satz 6, p. 260]. □

Claim 6.6. *For $\varepsilon \in (0, \frac{1}{48})$, assume that*

$$(\textbf{LAB}) \qquad 4 \le L < x^{\varepsilon}, \quad 0 < \alpha < \tfrac{1}{2}, \quad 0 < \beta \le \tfrac{1}{29} - \tfrac{48}{29}\varepsilon, \quad \tfrac{1}{3} + \tfrac{5}{6}\beta - \varepsilon > \alpha.$$

We then have

$$\#\mathscr{E}^{(6)} \ll 2^{\frac{a}{2}} x^{\frac{1}{2}-\varepsilon},$$

where the implicit constant depends on ε.

Proof. Since

$$\#\mathscr{E}^{(6)} \le \#\{(M,N) \in \mathscr{B}^{(5)} \mid p > x^{\frac{1}{2}-\beta}\} + \#\{(M,N) \in \mathscr{B}^{(5)} \mid q > x^{\frac{1}{2}-\beta}\},$$

by writing

$$\mathscr{E}^{(6)}_{\mathsf{p}} := \{(M,N) \in \mathscr{B}^{(5)} \mid p > x^{\frac{1}{2}-\beta}\},$$
$$\mathscr{E}^{(6)}_{\mathsf{q}} := \{(M,N) \in \mathscr{B}^{(5)} \mid q > x^{\frac{1}{2}-\beta}\},$$

it suffices to show

$$\#\mathscr{E}^{(6)}_{\mathsf{p}},\ \#\mathscr{E}^{(6)}_{\mathsf{q}} \ll 2^{\frac{a}{2}} x^{\frac{1}{2}-\varepsilon}. \tag{6.10}$$

We first prove (6.10) for $\#\mathscr{E}^{(6)}_{\mathsf{p}}$. Take $(M,N) \in \mathscr{E}^{(6)}_{\mathsf{p}}$ arbitrarily. By (6.2),

$$m = M/p < x^{\frac{1}{2}}/(x^{\frac{1}{2}-\beta}) = x^{\beta}$$

so by (C5),

$$n = mn/m > (x^{\frac{1}{2}}L^{-1})/x^{\beta} = x^{\frac{1}{2}-\beta}L^{-1},$$

which further implies by (6.2),

$$q = N/n < x^{\frac{1}{2}}/(x^{\frac{1}{2}-\beta}L^{-1}) = x^{\beta}L. \tag{6.11}$$

We then write N as the product of primes

$$N = q_1 q_2 \cdots q_r \quad \text{with} \quad q = q_1 \ge q_2 \ge \cdots \ge q_r.$$

By (C1) and (**LAB**), we can take the smallest index $i \in \{1, \ldots, r\}$ such that

$$q_1 \cdots q_i > x^{\frac{1}{6}-\frac{5}{6}\beta}.$$

Let $d := q_1 \cdots q_i$ and write $N = d\nu$. By the definition of i,

$$q_1 \cdots q_{i-1} \le x^{\frac{1}{6}-\frac{5}{6}\beta}. \tag{6.12}$$

Then, by (6.11), we have

$$x^{\frac{1}{6}-\frac{5}{6}\beta} < d = q_1 \cdots q_i = (q_1 \cdots q_{i-1}) \cdot q_i < x^{\frac{1}{6}+\frac{1}{6}\beta}L.$$

By (C1) and (6.12), we have

$$q_i q_{i+1} \cdots q_r = N/(q_1 \cdots q_{i-1}) > (x^{\frac{1}{2}}L^{-1})/(x^{\frac{1}{6}-\frac{5}{6}\beta}) = x^{\frac{1}{3}+\frac{5}{6}\beta}L^{-1}.$$

Thus, by (C2), $L < x^{\varepsilon}$ and (**LAB**), we find that

$$q_i > L^4. \tag{6.13}$$

By (C3), this implies $q_1, \ldots, q_i \parallel N$ so that

$$(d, 2\nu) = 1 \quad \text{and} \quad d\colon \text{square-free}. \tag{6.14}$$

Since $(M,N) \in \mathscr{B}$, by the definition of amicable pairs, we obtain

$$\sigma(d^2\nu^2) = \sigma(2^a p^2 m^2), \quad \sigma(d^2\nu^2) = 2^a p^2 m^2 + d^2\nu^2.$$

By (6.14), these imply

$$\sigma(d^2) \mid \sigma(2^a p^2 m^2) = p^2\sigma(2^a m^2) + p\sigma(2^a m^2) + \sigma(2^a m^2), \tag{6.15}$$

$$d^2\nu^2 \equiv -2^a p^2 m^2 \pmod{\sigma(d^2)}. \tag{6.16}$$

By multiplying both sides of (6.16) by $\sigma(2^a m^2)$,

$$d^2\sigma(2^a m^2)\nu^2 \equiv -p^2\sigma(2^a m^2)2^a m^2 \pmod{\sigma(d^2)}. \tag{6.17}$$

By (6.15), this implies

$$d^2\sigma(2^a m^2)\nu^2 \equiv p\sigma(2^a m^2)2^a m^2 + \sigma(2^a m^2)2^a m^2 \pmod{\sigma(d^2)},$$

or, equivalently,

$$d^2\sigma(2^a m^2)\nu^2 - \sigma(2^a m^2)2^a m^2 \equiv p\sigma(2^a m^2)2^a m^2 \pmod{\sigma(d^2)}.$$

By taking the square,

$$(d^2\sigma(2^a m^2)\nu^2 - \sigma(2^a m^2)2^a m^2)^2 \equiv p^2\sigma(2^a m^2)2^a m^2\sigma(2^a m^2)2^a m^2 \pmod{\sigma(d^2)}.$$

By (6.17), this can be rewritten as

$$(d^2\sigma(2^a m^2)\nu^2 - \sigma(2^a m^2)2^a m^2)^2 + d^2\sigma(2^a m^2)^2 2^a m^2\nu^2 \equiv 0 \pmod{\sigma(d^2)}. \tag{6.18}$$

We may regard this as a quartic congruence of the indeterminate ν. Note that this quartic congruence depends only on d, m and a. Let $\mathscr{N}(d,m,a)$ be the set of the solutions $(\mathrm{mod}\ \sigma(d^2))$ of the congruence (6.18). Then, by Lemma 6.5 with $k = 4$ and focusing on the leading coefficient $d^4\sigma(2^a m^2)^2$, we have

$$\#\mathscr{N}(d,m,a) \ll \sigma(d^2)^{\frac{3}{4}+\frac{\varepsilon}{3}}(d^4\sigma(2^a m^2)^2, \sigma(d^2))^{\frac{1}{4}} \le \sigma(d^2)^{\frac{3}{4}+\frac{\varepsilon}{3}}\sigma(2^a m^2)^{\frac{1}{2}}(d^4, \sigma(d^2))^{\frac{1}{4}}. \tag{6.19}$$

Recall that d is a product of primes $> L^4$ as shown in (6.13). Also, we have

$$(d^2, \sigma(d^2)) \mid (N^2, \sigma(N^2)) = (N^2, s(N^2)) = (N^2, 2^a M^2) = (N^2, M^2).$$

By these two observation and by (C4), we find that $(d^4, \sigma(d^2)) = 1$. Thus, by (6.19),

$$\#\mathscr{N}(d,m,a) \ll \sigma(d^2)^{\frac{3}{4}+\frac{\varepsilon}{3}}\sigma(2^a m^2)^{\frac{1}{2}} \ll 2^{\frac{a}{2}}x^{\frac{\varepsilon}{2}}d^{\frac{3}{2}}m \tag{6.20}$$

By the above argument, for any pair $(M,N) \in \mathscr{E}_{\mathsf{p}}^{(6)}$, we may write $N = d\nu$ with restrictions

$$m < x^\beta, \quad D_1 < d \le D_2, \quad \nu \ (\mathrm{mod}\ \sigma(d^2)) \in \mathscr{N}(d,m,a),$$

where

$$D_1 := x^{\frac{1}{6}-\frac{5}{6}\beta} \quad \text{and} \quad D_2 := x^{\frac{1}{6}+\frac{1}{6}\beta}L. \tag{6.21}$$

Therefore, since M is determined by N,

$$\begin{aligned}
\#\mathscr{E}_{\mathsf{p}}^{(6)} &\le \#\{(M,N) \in \mathscr{B}^{(5)} \mid m < x^\beta,\ D_1 < d \le D_2,\ \nu \ (\mathrm{mod}\ \sigma(d^2)) \in \mathscr{N}(d,m,a)\} \\
&\le \sum_{m<x^\beta}\ \sum_{D_1<d\le D_2}\ \sum_{\substack{\nu \le x^{\frac{1}{2}}/d \\ \nu\ (\mathrm{mod}\ \sigma(d^2)) \in \mathscr{N}(d,m,a)}} 1 \\
&= \sum_{m<x^\beta}\ \sum_{D_1<d\le D_2}\ \sum_{\rho\ (\mathrm{mod}\ \sigma(d^2)) \in \mathscr{N}(d,m,a)}\ \sum_{\substack{\nu \le x^{\frac{1}{2}}/d \\ \nu \equiv \rho\ (\mathrm{mod}\ \sigma(d^2))}} 1
\end{aligned}$$

$$\ll \sum_{m<x^\beta} \sum_{D_1<d\le D_2} \sum_{\rho \,(\mathrm{mod}\ \sigma(d^2))\in\mathcal{N}(d,m,a)} \left(1+\frac{x^{\frac{1}{2}}}{d\sigma(d^2)}\right).$$

By (6.20) and (6.21), this is

$$\ll 2^{\frac{a}{2}}x^{\frac{\varepsilon}{2}} \sum_{m<x^\beta} \sum_{D_1<d\le D_2} d^{\frac{3}{2}} m\left(1+\frac{x^{\frac{1}{2}}}{d\sigma(d^2)}\right) \ll 2^{\frac{a}{2}}x^{\frac{\varepsilon}{2}} \sum_{m<x^\beta} m \sum_{D_1<d\le D_2}\left(d^{\frac{3}{2}}+\frac{x^{\frac{1}{2}}}{d^{\frac{3}{2}}}\right)$$
$$\ll 2^{\frac{a}{2}}x^{2\beta+\frac{\varepsilon}{2}}(D_2^{\frac{5}{2}}+x^{\frac{1}{2}}D_1^{-\frac{1}{2}}).$$

By recalling the assumption $L\le x^\varepsilon$, (6.21) and (**LAB**), this is

$$\ll 2^{\frac{a}{2}}x^{\frac{\varepsilon}{2}}(x^{\frac{5}{12}+\frac{29}{12}\beta}L^{\frac{5}{2}}+x^{\frac{5}{12}+\frac{29}{12}\beta}) < 2^{\frac{a}{2}}x^{\frac{1}{2}-\varepsilon}.$$

This completes the estimate for (6.10) for $\mathscr{E}_{\mathsf{p}}^{(6)}$.

We next prove (6.10) for $\mathscr{E}_{\mathsf{q}}^{(6)}$. Take $(M,N)\in\mathscr{E}_{\mathsf{q}}^{(6)}$ arbitrarily. By (6.2),

$$n=N/q<x^{\frac{1}{2}}/(x^{\frac{1}{2}-\beta})=x^\beta$$

so by (C5),

$$m=mn/n>(x^{\frac{1}{2}}L^{-1})/x^\beta=x^{\frac{1}{2}-\beta}L^{-1},$$

which further implies by (6.2),

$$p=M/m<x^{\frac{1}{2}}/(x^{\frac{1}{2}-\beta}L^{-1})=x^\beta L. \tag{6.22}$$

We then write M as the product of primes

$$M=p_1p_2\cdots p_r \quad \text{with} \quad p=p_1\ge p_2\ge\cdots\ge p_r.$$

By (C1) and (**LAB**), we can take the smallest index i such that

$$p_1\cdots p_i>x^{\frac{1}{6}-\frac{5}{6}\beta}.$$

Let $d=p_1\cdots p_i$ and write $M=d\mu$. By the definition of i,

$$p_1\cdots p_{i-1}\le x^{\frac{1}{6}-\frac{5}{6}\beta}. \tag{6.23}$$

Then, by (6.22), we have

$$x^{\frac{1}{6}-\frac{5}{6}\beta}<d=p_1\cdots p_i=(p_1\cdots p_{i-1})\cdot p_i<x^{\frac{1}{6}+\frac{1}{6}\beta}L.$$

By (C1) and (6.23), this gives

$$p_ip_{i+1}\cdots p_r=M/(p_1\cdots p_{i-1})>(x^{\frac{1}{2}}L^{-1})/(x^{\frac{1}{6}-\frac{5}{6}\beta})=x^{\frac{1}{3}+\frac{5}{6}\beta}L^{-1}.$$

Thus, by (C2), $L<x^\varepsilon$ and (**LAB**), we find that

$$p_i>L^4. \tag{6.24}$$

By (C3), this implies $p_1,\dots,p_i\parallel M$ so that

$$(d,2\mu)=1 \quad \text{and} \quad d\text{: square-free}. \tag{6.25}$$

Since $(M,N)\in\mathscr{B}$, by the definition of amicable pairs, we obtain

$$\sigma(2^ad^2\mu^2)=\sigma(q^2n^2),\quad \sigma(2^ad^2\mu^2)=2^ad^2\mu^2+q^2n^2.$$

By (6.25), these imply

$$\sigma(d^2)\mid\sigma(q^2n^2)=q^2\sigma(n^2)+q\sigma(n^2)+\sigma(n^2), \tag{6.26}$$

$$2^a d^2 \mu^2 \equiv -q^2 n^2 \pmod{\sigma(d^2)}. \tag{6.27}$$

By multiplying $\sigma(n^2)$ to (6.27),

$$2^a d^2 \sigma(n^2) \mu^2 \equiv -q^2 \sigma(n^2) n^2 \pmod{\sigma(d^2)}. \tag{6.28}$$

By (6.26), this implies

$$2^a d^2 \sigma(n^2)\mu^2 \equiv q\sigma(n^2)n^2 + \sigma(n^2)n^2 \pmod{\sigma(d^2)},$$

or, equivalently,

$$2^a d^2 \sigma(n^2)\mu^2 - \sigma(n^2)n^2 \equiv q\sigma(n^2)n^2 \pmod{\sigma(d^2)}$$

By taking the square,

$$(2^a d^2 \sigma(n^2)\mu^2 - \sigma(n^2)n^2)^2 \equiv q^2\sigma(n^2)n^2 \cdot \sigma(n^2)n^2 \pmod{\sigma(d^2)}.$$

By (6.28), this can be rewritten as

$$(2^a d^2 \sigma(n^2)\mu^2 - \sigma(n^2)n^2)^2 + 2^a d^2 \sigma(n^2)^2 n^2 \mu^2 \equiv 0 \pmod{\sigma(d^2)}. \tag{6.29}$$

We may regard this as a quartic congruence of the indeterminate μ. Note that this quartic congruence depends only on d, n and a. Let $\mathscr{M}(d,n,a)$ be the set of the solutions (mod $\sigma(d^2)$) of the congruence (6.29). Then, by Lemma 6.5 with $k = 4$ and focusing on the leading coefficient $4^a d^4 \sigma(n^2)^2$, we have

$$\#\mathscr{M}(d,n,a) \ll \sigma(d^2)^{\frac{3}{4}+\frac{\varepsilon}{3}} (4^a d^4 \sigma(n^2)^2, \sigma(d^2))^{\frac{1}{4}} \le \sigma(d^2)^{\frac{3}{4}+\frac{\varepsilon}{3}} \sigma(n^2)^{\frac{1}{2}} (d^4, \sigma(d^2))^{\frac{1}{4}} \tag{6.30}$$

since $\sigma(d^2)$ is odd. Recall that d is a product of primes $> L^4$ as shown in (6.24). Also, we have

$$(d^2, \sigma(d^2)) \mid (M^2, \sigma(2^a M^2)) = (M^2, s(2^a M^2)) = (M^2, N^2).$$

By these two observation and by (C4), we find that $(d^4, \sigma(d^2)) = 1$. Thus, by (6.30),

$$\#\mathscr{M}(d,n,a) \ll \sigma(d^2)^{\frac{3}{4}+\frac{\varepsilon}{3}} \sigma(n^2)^{\frac{1}{2}} \ll x^{\frac{\varepsilon}{2}} d^{\frac{3}{2}} n \tag{6.31}$$

By the above argument, for any pair $(M,N) \in \mathscr{E}_{\mathsf{q}}^{(6)}$, we may write $m = d\mu$ with restrictions

$$n < x^\beta, \quad D_1 < d \le D_2, \quad \mu \pmod{\sigma(d^2)} \in \mathscr{M}(d,n,a),$$

where D_1, D_2 are defined as in (6.21). Therefore, since N is determined by M,

$$\begin{aligned}
\#\mathscr{E}_{\mathsf{q}}^{(6)} &\le \#\{(M,N) \in \mathscr{B}^{(5)} \mid n < x^\beta,\ D_1 < d \le D_2,\ \mu \pmod{\sigma(d^2)} \in \mathscr{M}(d,n,a)\} \\
&\le \sum_{n < x^\beta} \sum_{D_1 < d \le D_2} \sum_{\substack{\mu \le x^{\frac{1}{2}}/d \\ \mu \pmod{\sigma(d^2)} \in \mathscr{M}(d,n,a)}} 1 \\
&= \sum_{n < x^\beta} \sum_{D_1 < d \le D_2} \sum_{\rho \pmod{\sigma(d^2)} \in \mathscr{M}(d,n,a)} \sum_{\substack{\mu \le x^{\frac{1}{2}}/d \\ \mu \equiv \rho \pmod{\sigma(d^2)}}} 1 \\
&\ll \sum_{n < x^\beta} \sum_{D_1 < d \le D_2} \sum_{\rho \pmod{\sigma(d^2)} \in \mathscr{M}(d,n,a)} \left(1 + \frac{x^{\frac{1}{2}}}{d\sigma(d^2)}\right).
\end{aligned}$$

By (6.31), this is

$$\ll X^{\frac{\varepsilon}{2}} \sum_{n < X^\beta} \sum_{D_1 < d \le D_2} d^{\frac{3}{2}} n \left(1 + \frac{X^{\frac{1}{2}}}{d\sigma(d^2)}\right) \ll x^{\frac{\varepsilon}{2}} \sum_{n < x^\beta} n \sum_{D_1 < d \le D_2} \left(d^{\frac{3}{2}} + \frac{x^{\frac{1}{2}}}{d^{\frac{3}{2}}}\right)$$

$$\ll x^{2\beta+\frac{\varepsilon}{2}}(D_2^{\frac{5}{2}} + x^{\frac{1}{2}}D_1^{-\frac{1}{2}}).$$

By recalling the assumption $0 < \beta < \frac{1}{29} - \frac{48}{29}\varepsilon$, $L \le x^{\varepsilon}$, (6.21) and (**LAB**), this is

$$\ll x^{\frac{\varepsilon}{2}}\big(x^{\frac{5}{12}+\frac{29}{12}\beta}L^{\frac{5}{2}} + x^{\frac{5}{12}+\frac{29}{12}\beta}\big) < x^{\frac{1}{2}-\varepsilon}.$$

This completes the proof. □

From now on, we always assume (**LAB**). For a positive integer m, let

$$D^{\flat}(m) \coloneqq \prod_{p\|m} p,$$

which is "the square-free part" of m. Note that we have

$$m = D^{\flat}(m)D^{\sharp}(m) \quad \text{and} \quad (D^{\flat}(m), D^{\sharp}(m)) = 1$$

for any positive integer m and the first decomposition is unique.

As our final restriction, we introduce

$$\text{(C7)} \qquad \min(p_{\max}(\sigma(D^{\flat}(m)^2)), p_{\max}(\sigma(D^{\flat}(n)^2))) > K,$$

where $K \coloneqq L^4 + L^2 + 1$ and we used the factorization given in (6.4). Let

$$\mathscr{B}^{(6)} = \mathscr{B}^{(7)} \sqcup \mathscr{E}^{(7)},$$

where

$$\mathscr{B}^{(7)} \coloneqq \{(M,N) \in \mathscr{B}^{(6)} \mid \min(p_{\max}(\sigma(D^{\flat}(m)^2)), p_{\max}(\sigma(D^{\flat}(n)^2))) > K\},$$
$$\mathscr{E}^{(7)} \coloneqq \{(M,N) \in \mathscr{B}^{(6)} \mid \min(p_{\max}(\sigma(D^{\flat}(m)^2)), p_{\max}(\sigma(D^{\flat}(n)^2))) \le K\}.$$

Claim 6.7. *We have*

$$\#\mathscr{E}^{(7)} \ll x^{\frac{1}{2}}(\log x)^2 \exp\big(-\tfrac{\beta}{4}u \log\log u\big)$$

for large u, where the implicit constant is absolute.

Proof. As before, it suffices to bound the cardinalities of

$$\mathscr{E}^{(7)}_{\mathsf{m}} \coloneqq \{(M,N) \in \mathscr{B}^{(6)} \mid p_{\max}(\sigma(D^{\flat}(m)^2)) \le K\}, \tag{6.32}$$

$$\mathscr{E}^{(7)}_{\mathsf{n}} \coloneqq \{(M,N) \in \mathscr{B}^{(6)} \mid p_{\max}(\sigma(D^{\flat}(n)^2)) \le K\}. \tag{6.33}$$

We first bound $\mathscr{E}^{(7)}_{\mathsf{m}}$. For $(M,N) \in \mathscr{E}^{(7)}_{\mathsf{m}}$, by (C3) and (C6), we have

$$p \le x^{\frac{1}{2}-\beta} \quad \text{and} \quad D^{\sharp}(m) \le L^2.$$

Thus, by recalling (6.2) and writing $d \coloneqq D^{\sharp}(m)$ and $\mu \coloneqq D^{\flat}(m)$,

$$\#\mathscr{E}^{(7)}_{\mathsf{m}} \le \sum_{p \le x^{\frac{1}{2}-\beta}} \sum_{d \le L^2} \sum_{\substack{\mu \le x^{\frac{1}{2}}/(pd) \\ \mu:\text{square-free} \\ p_{\max}(\sigma(\mu^2)) \le K}} 1 \tag{6.34}$$

We apply Theorem 1.2 to the last sum. By (**L**) we have

$$L^4 + L^2 + 1 \ge \log \frac{x^{\frac{1}{2}}}{pd} \log\log \frac{x^{\frac{1}{2}}}{pd}$$

so the assumption of Theorem 1.2 is satisfied. For $p \le x^{\frac{1}{2}-\beta}$ and $d \le L^2$, we have

$$\frac{\log \frac{x^{\frac12}}{pd}}{\log K} \ge \frac{\log(x^\beta L^{-2})}{\log(L^4+L^2+1)} = \frac{\log(x^\beta L^{-2})}{\log L^4} - \frac{\log(x^\beta L^{-2})\log(1+\frac{L^2+1}{L^4})}{(\log(L^4+L^2+1))(\log L^4)}$$
$$\ge \frac{\beta}{4}u - \frac12 - \frac{\beta\log x}{L^2(\log L)^2} \ge \frac{\beta}{4}u - 1$$

for large u since $L \ge \log x \ge u$ by (**L**). Thus, using Theorem 1.2 in (6.34) and noting

$$\left(\frac{\beta}{4}u-1\right)\log\log\left(\frac{\beta}{4}u-1+e\right) + \left(\frac{\beta}{4}u-1\right)\log\log\left(\log\frac{\beta}{4}u+e\right) + O(u)$$
$$\ge \frac{\beta}{4}u\log\log\frac{\beta}{4}u + \frac{\beta}{4}u\log\log\log\frac{\beta}{4}u + O(u) \ge \frac{\beta}{4}u\log\log u$$

for large u, we have

$$\#\mathscr{E}_{\mathsf{m}}^{(7)} \le x^{\frac12}\exp\left(-\tfrac{\beta}{4}u\log\log u\right)\sum_{p\le x^{\frac12-\beta}}\sum_{d\le L^2}\frac{1}{pd} \ll x^{\frac12}(\log x)^2\exp\left(-\tfrac{\beta}{4}u\log\log u\right) \tag{6.35}$$

for large u. This completes the bound for (6.32).

We next bound $\mathscr{E}_{\mathsf{n}}^{(7)}$. For $(M,N)\in\mathscr{B}^{(6)}$, by (C3) and (C6), we have

$$q \le x^{\frac12-\beta} \quad\text{and}\quad D^\sharp(n)\le L^2.$$

Thus, by recalling (6.2) and writing $d := D^\sharp(n)$ and $\nu := D^\flat(n)$,

$$\#\mathscr{E}_{\mathsf{n}}^{(7)} \le \sum_{q\le x^{\frac12-\beta}}\sum_{d\le L^2}\sum_{\substack{\nu\le x^{\frac12}/(qd)\\ \nu:\text{square-free}\\ p_{\max}(\sigma(\nu^2))\le K}} 1.$$

This sum can be estimated similarly to (6.35). This completes the bound for (6.33). □

Finally, we can estimate the remaining part without adding further restriction. For this estimate, we prepare the following preliminary estimate.

Lemma 6.6. *For $x, L\ge 1$, we have*

$$\sum_{P>L^4}\frac{1}{P^{\frac12}}\sum_{\substack{L^2<Q\le x^{\frac12}\\ P\mid\sigma(Q^2)}}\frac{1}{Q} \ll \frac{\log x}{L},$$

where P, Q runs through prime numbers and the implicit constant is absolute.

Proof. We first decompose the sum as

$$\sum_{P>L^4}\frac{1}{P^{\frac12}}\sum_{\substack{L^2<Q\le x^{\frac12}\\ P\mid\sigma(Q^2)}}\frac{1}{Q} = \sum\nolimits_1 + \sum\nolimits_2, \tag{6.36}$$

where

$$\sum\nolimits_1 := \sum_{P>L^4}\frac{1}{P^{\frac12}}\sum_{\substack{L^2<Q\le\min(P,x^{\frac12})\\ P\mid\sigma(Q^2)}}\frac{1}{Q} \quad\text{and}\quad \sum\nolimits_2 := \sum_{P>L^4}\frac{1}{P^{\frac12}}\sum_{\substack{P<Q\le x^{\frac12}\\ P\mid\sigma(Q^2)}}\frac{1}{Q}.$$

The sum $\sum_1$ can be estimated by using $\omega(n) \ll 1 + \log n$ as

$$\sum\nolimits_1 \leq \sum_{L^2 < Q \leq x^{\frac{1}{2}}} \frac{1}{Q} \sum_{\substack{P \mid \sigma(Q^2) \\ Q \leq P}} \frac{1}{P^{\frac{1}{2}}} \ll \sum_{L^2 < Q \leq x^{\frac{1}{2}}} \frac{\omega(\sigma(Q^2))}{Q^{\frac{3}{2}}} \ll \frac{\log x}{L}. \tag{6.37}$$

On the other hand, for the sum $\sum_2$, we dissect the inner sum as

$$\begin{aligned} \sum_{\substack{P < Q \leq x^{\frac{1}{2}} \\ P \mid \sigma(Q^2)}} \frac{1}{Q} &\ll (\log x) \sup_{P \leq U \leq x^{\frac{1}{2}}} U^{-1} \sum_{\substack{U < Q \leq 2U \\ Q^2 + Q + 1 \equiv 0 \ (\mathrm{mod}\ P)}} 1 \\ &\ll (\log x) \sup_{P \leq U \leq x^{\frac{1}{2}}} U^{-1} \left(\frac{U}{P} + 1 \right) \ll \frac{\log x}{P}. \end{aligned}$$

Therefore, $\sum_2$ can be bounded by

$$\sum\nolimits_2 \ll \sum_{P > L^4} \frac{\log x}{P^{\frac{3}{2}}} \ll \frac{\log x}{L^2}. \tag{6.38}$$

By combining (6.36), (6.37) and (6.38), we obtain the lemma. □

Claim 6.8. *We have*

$$\#\mathscr{B}^{(7)} \ll x^{\frac{1}{2}} (\log x)^6 L^{-1},$$

where the implicit constant is absolute.

Proof. By (6.4) and (C4), we find that $p \neq q$ for any $(M, N) \in \mathscr{B}^{(7)}$. Thus,

$$\mathscr{B}^{(7)} = \mathscr{B}_{\mathsf{p}}^{(7)} \sqcup \mathscr{B}_{\mathsf{q}}^{(7)},$$

where

$$\mathscr{B}_{\mathsf{p}}^{(7)} := \{(M, N) \in \mathscr{B}^{(7)} \mid p > q\} \quad \text{and} \quad \mathscr{B}_{\mathsf{q}}^{(7)} := \{(M, N) \in \mathscr{B}^{(7)} \mid p < q\}.$$

It suffices to bound the cardinalities of these sets separately.

We begin with $\mathscr{B}_{\mathsf{p}}^{(7)}$. Take $(M, N) \in \mathscr{B}_{\mathsf{p}}^{(7)}$ arbitrarily. By (C7), there is a prime P satisfying

$$P > L^4 + L^2 + 1 \quad \text{and} \quad P \mid \sigma(D^\flat(m)^2). \tag{6.39}$$

Then, since $D^\flat(m)$ is square-free, there is a prime number Q satisfying

$$Q \parallel D^\flat(m) \quad \text{and} \quad P \mid \sigma(Q^2). \tag{6.40}$$

By (6.39) and (6.40), we find that

$$L^4 + L^2 + 1 < P \leq \sigma(Q^2) = Q^2 + Q + 1$$

so that by (6.40)

$$L^2 < Q \leq x^{\frac{1}{2}} \quad \text{and} \quad P \mid \sigma(Q^2). \tag{6.41}$$

By (6.40) and recalling that $(2^a M^2, N^2)$ is amicable, we also find that

$$P \mid \sigma(Q^2) \mid \sigma(2^a M^2) = \sigma(N^2).$$

Thus, there should be a prime number R and a positive integer e such that

$$R^e \parallel N, \quad R^e \leq x^{\frac{1}{2}} \quad \text{and} \quad P \mid \sigma(R^{2e}). \tag{6.42}$$

Note that by (6.41) and (C4), we have

$$Q \neq R. \tag{6.43}$$

Since $(2^a M^2, N^2)$ is an amicable pair, we have

$$N^2 = \sigma(2^a p^2 m^2) - 2^a p^2 m^2 = p^2 s(2^a m^2) + p\sigma(2^a m^2) + \sigma(2^a m^2). \tag{6.44}$$

Let

$$\tilde{e} := \begin{cases} 1 & \text{if } e = 1, \\ 2e & \text{if } e \geq 2. \end{cases} \tag{6.45}$$

By taking the reduction (mod $R^{\tilde{e}}$) of (6.44) and recalling (6.42),

$$p^2 s(2^a m^2) + p\sigma(2^a m^2) + \sigma(2^a m^2) \equiv 0 \pmod{R^{\tilde{e}}}. \tag{6.46}$$

Define a non-negative integer f by

$$R^f := (s(2^a m^2), \sigma(2^a m^2), R^{\tilde{e}}).$$

Obviously,

$$f \leq \tilde{e} \leq \log x \tag{6.47}$$

Since $2^a m^2 = \sigma(2^a m^2) - s(2^a m^2)$ and R is odd, we have $R^f \mid m^2$ and so

$$R^{\lceil \frac{f}{2} \rceil} \mid m.$$

By (6.40) and (6.43), this implies

$$QR^{\lceil \frac{f}{2} \rceil} \mid m. \tag{6.48}$$

Let $\mathscr{P}(a, m, R^{\tilde{e}})$ be the set of the solutions of the quadratic congruence (6.46) in the indeterminate p. Then, (6.46) can be rephrased as

$$p \pmod{R^{\tilde{e}}} \in \mathscr{P}(a, m, R^{\tilde{e}}). \tag{6.49}$$

By Lemma 6.5 and (6.47), the number of solutions $\#\mathscr{P}(a, m, R^{\tilde{e}})$ can be bounded by

$$\#\mathscr{P}(a, m, R^{\tilde{e}}) \leq R^{\frac{f}{2}} R^{\frac{\tilde{e}}{2}} \tau(R^{\tilde{e}}) = (\tilde{e}+1) R^{\frac{\tilde{e}+f}{2}} \ll (\log x) R^{\frac{\tilde{e}+f}{2}}. \tag{6.50}$$

If $e = 1$, then (6.46) becomes a quadratic equation in the finite field $\mathbb{F}_R$. Therefore, if $f = 0$, then $\#\mathscr{P}(a, m, R^{\tilde{e}}) \leq 2 = 2R^{\lceil \frac{f}{2} \rceil}$ and if $f = 1$, then $\#\mathscr{P}(a, m, R^{\tilde{e}}) = R = R^{\lceil \frac{f}{2} \rceil}$. Thus,

$$e = 1 \implies \#\mathscr{P}(a, m, R^{\tilde{e}}) \ll R^{\lceil \frac{f}{2} \rceil}. \tag{6.51}$$

Furthermore, since we have $p > q$ for $\mathscr{B}_{\mathsf{p}}^{(7)}$, we find that

$$p > q = p_{\max}(N) \geq R.$$

Thus, $p > R^e$ if $e = 1$. If $e \geq 2$, then by (6.4) and (C3), $p > L^4 \geq R^{2e}$. In any case,

$$p > R^{\tilde{e}}. \tag{6.52}$$

By the above argument, for any $(M, N) \in \mathscr{B}_{\mathsf{p}}^{(7)}$, we can find

P, Q, R: prime numbers, e: a positive integer, f: a non-negative integer

satisfying (6.39), (6.41), (6.42), (6.47) for which the decomposition

$$M = pm \leq x^{\frac{1}{2}}$$

satisfies (6.48), (6.49), (6.52). Since for any $(M,N) \in \mathscr{B}_{\mathsf{p}}^{(7)}$, N is uniquely determined by M, the above observations give

$$\#\mathscr{B}_{\mathsf{p}}^{(7)} \leq \sum_{P>L^4} \sum_{\substack{L^2<Q\leq x^{\frac{1}{2}} \\ P\mid\sigma(Q^2)}} \sum_{\substack{R^e\leq x^{\frac{1}{2}} \\ P\mid\sigma(R^{2e})}} \sum_{0\leq f\leq \tilde{e}} \sum_{\substack{m\leq x^{\frac{1}{2}}/R^{\tilde{e}} \\ QR^{\lceil \frac{f}{2}\rceil}\mid m}} \sum_{\substack{R^{\tilde{e}}<p\leq x^{\frac{1}{2}}/m \\ p \,(\mathrm{mod}\ R^{\tilde{e}})\in\mathscr{P}(a,m,R^{\tilde{e}})}} 1. \tag{6.53}$$

We consider the two most inner sums

$$\sum_{\substack{m\leq x^{\frac{1}{2}}/R^{\tilde{e}} \\ QR^{\lceil \frac{f}{2}\rceil}\mid m}} \sum_{\substack{R^{\tilde{e}}<p\leq x^{\frac{1}{2}}/m \\ p \,(\mathrm{mod}\ R^{\tilde{e}})\in\mathscr{P}(a,m,R^{\tilde{e}})}} 1. \tag{6.54}$$

By the condition $m \leq x^{\frac{1}{2}}/R^{\tilde{e}}$, we can bound the inner most sum by

$$\sum_{\substack{R^{\tilde{e}}<p\leq x^{\frac{1}{2}}/m \\ p \,(\mathrm{mod}\ R^{\tilde{e}})\in\mathscr{P}(a,m,R^{\tilde{e}})}} 1 = \sum_{\rho\in\mathscr{P}(a,m,R^{\tilde{e}})} \sum_{\substack{R^{\tilde{e}}<p\leq x^{\frac{1}{2}}/m \\ p\equiv\rho \,(\mathrm{mod}\ R^{\tilde{e}})}} 1 \ll \#\mathscr{P}(a,m,R^{\tilde{e}})\frac{x^{\frac{1}{2}}}{mR^{\tilde{e}}}.$$

Therefore, if $e \geq 2$, then (6.54) is bounded by using (6.50) as

$$\sum_{\substack{m\leq x^{\frac{1}{2}}/R^{\tilde{e}} \\ QR^{\lceil \frac{f}{2}\rceil}\mid m}} \sum_{\substack{R^{\tilde{e}}<p\leq x^{\frac{1}{2}}/m \\ p \,(\mathrm{mod}\ R^{\tilde{e}})\in\mathscr{P}(a,m,R^{\tilde{e}})}} 1 \ll \sum_{\substack{m\leq x^{\frac{1}{2}}/R^{\tilde{e}} \\ QR^{\lceil \frac{f}{2}\rceil}\mid m}} \frac{x^{\frac{1}{2}}(\log x)}{mR^{\frac{\tilde{e}-f}{2}}} \ll \frac{x^{\frac{1}{2}}(\log x)^2}{QR^{\frac{\tilde{e}}{2}}} = \frac{x^{\frac{1}{2}}(\log x)^2}{QR^e}.$$

On the other hand, if $e = 1$, then (6.54) is bounded by using (6.51) as

$$\sum_{\substack{m\leq x^{\frac{1}{2}}/R^{\tilde{e}} \\ QR^{\lceil \frac{f}{2}\rceil}\mid m}} \sum_{\substack{R^{\tilde{e}}<p\leq x^{\frac{1}{2}}/m \\ p \,(\mathrm{mod}\ R^{\tilde{e}})\in\mathscr{P}(2^a m^2,R^{\tilde{e}})}} 1 \ll \sum_{\substack{m\leq x^{\frac{1}{2}}/R^{\tilde{e}} \\ QR^{\lceil \frac{f}{2}\rceil}\mid m}} \frac{x^{\frac{1}{2}}(\log x)}{mR^{\tilde{e}-\lceil \frac{f}{2}\rceil}} \ll \frac{x^{\frac{1}{2}}(\log x)^2}{QR^{\tilde{e}}} = \frac{x^{\frac{1}{2}}(\log x)^2}{QR^e}.$$

By substituting these estimates into (6.53) and using (6.47), we have

$$\#\mathscr{B}_{\mathsf{p}}^{(7)} \ll \sum_{P>L^4} \sum_{\substack{L^2<Q\leq x^{\frac{1}{2}} \\ P\mid\sigma(Q^2)}} \sum_{\substack{R^e\leq x^{\frac{1}{2}} \\ P\mid\sigma(R^{2e})}} \sum_{0\leq f\leq \tilde{e}} \frac{x^{\frac{1}{2}}(\log x)^2}{QR^e} \ll \sum_{P>L^4} \sum_{\substack{L^2<Q\leq x^{\frac{1}{2}} \\ P\mid\sigma(Q^2)}} \sum_{\substack{R^e\leq x^{\frac{1}{2}} \\ P\mid\sigma(R^{2e})}} \frac{x^{\frac{1}{2}}(\log x)^3}{QR^e}.$$

By using Lemma 6.3 to bound the most inner sum, we can continue the bound as

$$\#\mathscr{B}_{\mathsf{p}}^{(7)} \ll x^{\frac{1}{2}}(\log x)^5 \sum_{P>L^4} \frac{1}{P^{\frac{1}{2}}} \sum_{\substack{L^2<Q\leq x^{\frac{1}{2}} \\ P\mid\sigma(Q^2)}} \frac{1}{Q}.$$

By Lemma 6.6, we obtain

$$\#\mathscr{B}_{\mathsf{p}}^{(7)} \ll x^{\frac{1}{2}}(\log x)^6 L^{-1}.$$

This completes the estimate for $\mathscr{B}_{\mathsf{p}}^{(7)}$.

We next consider $\mathscr{B}_{\mathsf{q}}^{(7)}$. To avoid introducing new letters, we refresh the above notation for $\mathscr{B}_{\mathsf{p}}^{(7)}$. Take $(M,N) \in \mathscr{B}_{\mathsf{q}}^{(7)}$ arbitrarily. Then, by (C7), there is a prime number P satisfying

$$P > L^4 + L^2 + 1 \quad \text{and} \quad P \mid \sigma(D^\flat(n)^2). \tag{6.55}$$

Then, since $D^\flat(n)$ is square-free, there is a prime number Q satisfying

$$Q \parallel D^\flat(n) \quad \text{and} \quad P \mid \sigma(Q^2). \tag{6.56}$$

By (6.55) and (6.56), we find that

$$L^2 < Q \le x^{\frac{1}{2}} \quad \text{and} \quad P \mid \sigma(Q^2). \tag{6.57}$$

By (6.56), we also find that

$$P \mid \sigma(Q^2) \mid \sigma(N^2) = \sigma(2^a M^2).$$

Thus, there should be a prime number R and a positive integer e such that

$$R^{2e} \parallel 2^a M^2, \quad R^e \le x^{\frac{1}{2}} \quad \text{and} \quad P \mid \sigma(R^{2e}), \tag{6.58}$$

where $R = 2$ could be the case. Note that by (6.57) and (C4), we have

$$Q \neq R. \tag{6.59}$$

Since $(2^a M^2, N^2)$ is an amicable pair, we have

$$2^a M^2 = \sigma(q^2 n^2) - q^2 n^2 = q^2 s(n^2) + q\sigma(n^2) + \sigma(n^2). \tag{6.60}$$

Define $\widetilde{e}$ by (6.45) as before. By taking the reduction (mod $R^{\tilde{e}}$) of (6.60) and recalling (6.58),

$$q^2 s(n^2) + q\sigma(n^2) + \sigma(n^2) \equiv 0 \pmod{R^{\tilde{e}}}. \tag{6.61}$$

Define a non-negative integer f by

$$R^f \coloneqq (s(n^2), \sigma(n^2), R^{\tilde{e}}).$$

Obviously,

$$f \le \tilde{e} \ll \log x \tag{6.62}$$

Since $n^2 = \sigma(n^2) - s(n^2)$, we have $R^f \mid n^2$ so

$$R^{\lceil \frac{f}{2} \rceil} \mid n.$$

By (6.56) and (6.59), this implies

$$QR^{\lceil \frac{f}{2} \rceil} \mid n. \tag{6.63}$$

Let $\mathfrak{Q}(n, R^{\tilde{e}})$ be the set of the solutions of the quadratic congruence (6.61) in the indeterminate q. Then, (6.61) can be rephrased as

$$q \pmod{R^{\tilde{e}}} \in \mathfrak{Q}(n, R^{\tilde{e}}). \tag{6.64}$$

By Lemma 6.5 and (6.62), the number of solutions $\#\mathfrak{Q}(n, R^{\tilde{e}})$ can be bounded by

$$\#\mathfrak{Q}(n, R^{\tilde{e}}) \le R^{\frac{f}{2}} R^{\frac{\tilde{e}}{2}} \tau(R^{\tilde{e}}) = (\tilde{e}+1) R^{\frac{\tilde{e}+f}{2}} \ll (\log x) R^{\frac{\tilde{e}+f}{2}}. \tag{6.65}$$

If $e = 1$, then (6.61) becomes a quadratic equation in the finite field $\mathbb{F}_R$. Therefore, if $f = 0$, then $\#\mathfrak{Q}(n, R^{\tilde{e}}) \le 2 = 2R^{\lceil \frac{f}{2} \rceil}$ and if $f = 1$, then $\#\mathfrak{Q}(n, R^{\tilde{e}}) = R = R^{\lceil \frac{f}{2} \rceil}$. Thus,

$$e = 1 \implies \#\mathfrak{Q}(n, R^{\tilde{e}}) \ll R^{\lceil \frac{f}{2} \rceil}. \tag{6.66}$$

Furthermore, since we have $q > p$ for $\mathscr{B}_{\mathsf{q}}^{(7)}$, we find that

$$q > p = p_{\max}(M) \ge R.$$

Thus, $q > R^e$ if $e = 1$. If $e \ge 2$, then by (6.4) and (C3), $q > L^4 \ge R^{2e}$. In any case,

$$q > R^{\tilde{e}}. \tag{6.67}$$

By the above argument, for any $(M,N)\in\mathscr{B}_{\mathsf{q}}^{(7)}$, we can find

$$P,Q,R\colon \text{prime numbers},\quad e\colon \text{a positive integer},\quad f\colon \text{a non-negative integer}$$

satisfying (6.55), (6.57), (6.58), (6.62) for which the decomposition

$$N=qn\le x^{\frac12}$$

satisfies (6.63), (6.64), (6.67). Since for any $(M,N)\in\mathscr{B}_{\mathsf{q}}^{(7)}$, M is uniquely determined by N, the above observations give

$$\#\mathscr{B}_{\mathsf{q}}^{(7)}\le\sum_{P>L^4}\sum_{\substack{L^2<Q\le x^{\frac12}\\ P|\sigma(Q^2)}}\sum_{\substack{R^e\le x^{\frac12}\\ P|\sigma(R^{2e})}}\sum_{0\le f\le\tilde e}\sum_{\substack{n\le x^{\frac12}/R^{\tilde e}\\ QR^{\lceil\frac f2\rceil}|n}}\sum_{\substack{R^{\tilde e}<q\le x^{\frac12}/n\\ q\,(\mathrm{mod}\ R^{\tilde e})\in\mathscr{Q}(n,R^{\tilde e})}}1. \tag{6.68}$$

We consider the two most inner sums

$$\sum_{\substack{n\le x^{\frac12}/R^{\tilde e}\\ QR^{\lceil\frac f2\rceil}|n}}\sum_{\substack{R^{\tilde e}<q\le x^{\frac12}/n\\ q\,(\mathrm{mod}\ R^{\tilde e})\in\mathscr{Q}(n,R^{\tilde e})}}1. \tag{6.69}$$

By the condition $n\le x^{\frac12}/R^{\tilde e}$, we can bound the inner most sum by

$$\sum_{\substack{R^{\tilde e}<q\le x^{\frac12}/n\\ q\,(\mathrm{mod}\ R^{\tilde e})\in\mathscr{Q}(n,R^{\tilde e})}}1=\sum_{\rho\in\mathscr{Q}(n,R^{\tilde e})}\sum_{\substack{R^{\tilde e}<q\le x^{\frac12}/n\\ q\equiv\rho\,(\mathrm{mod}\ R^{\tilde e})}}1\ll\#\mathscr{Q}(n,R^{\tilde e})\frac{x^{\frac12}}{nR^{\tilde e}}.$$

Therefore, if $e\ge2$, then (6.69) is bounded by using (6.65) as

$$\sum_{\substack{n\le x^{\frac12}/R^{\tilde e}\\ QR^{\lceil\frac f2\rceil}|n}}\sum_{\substack{R^{\tilde e}<q\le x^{\frac12}/n\\ q\,(\mathrm{mod}\ R^{\tilde e})\in\mathscr{Q}(n,R^{\tilde e})}}1\ll\sum_{\substack{n\le x^{\frac12}/R^{\tilde e}\\ QR^{\lceil\frac f2\rceil}|n}}\frac{x^{\frac12}(\log x)}{nR^{\frac{\tilde e-f}{2}}}\ll\frac{x^{\frac12}(\log x)^2}{QR^{\frac{\tilde e}{2}}}=\frac{x^{\frac12}(\log x)^2}{QR^e}$$

On the other hand, if $e=1$, then (6.69) is bounded by using (6.66) as

$$\sum_{\substack{n\le x^{\frac12}/R^{\tilde e}\\ QR^{\lceil\frac f2\rceil}|n}}\sum_{\substack{R^{\tilde e}<q\le x^{\frac12}/n\\ q\,(\mathrm{mod}\ R^{\tilde e})\in\mathscr{Q}(n,R^{\tilde e})}}1\ll\sum_{\substack{n\le x^{\frac12}/R^{\tilde e}\\ QR^{\lceil\frac f2\rceil}|n}}\frac{x^{\frac12}(\log x)}{nR^{\tilde e-\lceil\frac f2\rceil}}\ll\frac{x^{\frac12}(\log x)^2}{QR^{\tilde e}}=\frac{x^{\frac12}(\log x)^2}{QR^e}$$

By substituting these estimates into (6.68), we have

$$\#\mathscr{B}_{\mathsf{q}}^{(7)}\ll\sum_{P>L^4}\sum_{\substack{L^2<Q\le x^{\frac12}\\ P|\sigma(Q^2)}}\sum_{\substack{R^e\le x^{\frac12}\\ P|\sigma(R^{2e})}}\sum_{0\le f\le\tilde e}\frac{x^{\frac12}(\log x)^2}{QR^e}.$$

Then, by the same argument as we used for $\mathscr{B}_{\mathsf{p}}^{(7)}$, we obtain

$$\#\mathscr{B}_{\mathsf{q}}^{(7)}\ll x^{\frac12}(\log x)^6L^{-1}.$$

This completes the estimate for $\mathscr{B}_{\mathsf{q}}^{(7)}$ and the proof of the lemma. □

We now start the completion of the proof. It suffices to prove

$$\begin{aligned}&\#\{(A,B)\in\mathbb{N}^2 \mid (A,B)\colon \text{amicable},\ A\not\equiv B \pmod 2),\ A,B\le x\}\\ &\le x^{\frac12}\exp\bigl(-(\tfrac{1}{2\sqrt{29}}+o(1))(\log x\log\log\log x)^{\frac12}\bigr)\end{aligned}\tag{6.70}$$

as $x\to\infty$ since if (A,B) is an amicable pair and $\min(A,B)\le x$, then

$$\max(A,B)=s(\min(A,B))\le\sigma(\min(A,B))\ll x\log x$$

and so by (6.70), we can conclude that

$$\begin{aligned}&\#\{(A,B)\in\mathbb{N}^2 \mid (A,B)\colon \text{amicable},\ A\not\equiv B \pmod 2),\ \min(A,B)\le x\}\\ &\le \#\{(A,B)\in\mathbb{N}^2 \mid (A,B)\colon \text{amicable},\ A\not\equiv B \pmod 2),\ A,B\ll x\log x\}\\ &\le (x\log x)^{\frac12}\exp\bigl(-(\tfrac{1}{2\sqrt{29}}+o(1))(\log x\log\log\log x)^{\frac12}\bigr)\\ &\le x^{\frac12}\exp\bigl(-(\tfrac{1}{2\sqrt{29}}+o(1))(\log x\log\log\log x)^{\frac12}\bigr)\end{aligned}$$

as $x\to\infty$.

By using Lemma 6.1 and recalling (6.2), we have

$$\begin{aligned}&\#\{(A,B)\in\mathbb{N}^2 \mid (A,B)\colon \text{amicable},\ A\not\equiv B \pmod 2),\ A,B\le x\}\\ &=\#\{(a,M,N)\in\mathbb{N}^3 \mid (2^aM^2,N^2)\colon \text{amicable},\ M,N:\text{odd},\ 2^aM^2,N^2\le x\}\\ &=\sum_{a=1}^{O(\log x)}\#\mathscr{B}(x,a)=\sum_{1\le a\le\frac{\log x}{\log\log x}}\#\mathscr{B}(x,a)+\sum_{\frac{\log x}{\log\log x}<a\ll\log x}\#\mathscr{B}(x,a).\end{aligned}$$

Since for $(M,N)\in\mathscr{B}(x,a)$, N is uniquely determined by a and M, we have

$$\#\mathscr{B}(x,a)\le\#\{M\in\mathbb{N}\mid M\le 2^{-\frac a2}x^{\frac12}\}\le 2^{-\frac a2}x^{\frac12}$$

so that

$$\sum_{\frac{\log x}{\log\log x}<a\ll\log X}\#\mathscr{B}(x,a)\le x^{\frac12}\sum_{\frac{\log x}{\log\log x}<a\ll\log x}2^{-\frac a2}\le x^{\frac12}\exp\biggl(-c\biggl(\frac{\log x}{\log\log x}\biggr)\biggr)$$

for some $c>0$. Thus, it suffices to show

$$\sum_{1\le a\le\frac{\log x}{\log\log x}}\#\mathscr{B}(x,a)\le x^{\frac12}\exp\bigl(-(\tfrac{1}{2\sqrt{29}}+o(1))(\log x\log\log\log x)^{\frac12}\bigr)$$

as $x\to\infty$ by using our preceding arguments.

Take $\varepsilon\in(0,\frac{1}{48})$ arbitrarily. Choose parameters L,α,β by

$$L\coloneqq\exp\bigl(\tfrac{1}{2\sqrt{29}}(\log x\log\log\log x)^{\frac12}\bigr),\quad \alpha\coloneqq\tfrac14,\quad \beta\coloneqq\tfrac{1}{29}-\tfrac{48}{29}\varepsilon.$$

Then, for sufficiently large x, these choices satisfy (**L**), (**LA**) and (**LAB**) so these choices are available in our preceding arguments. Also, we have

$$u=\frac{\log x}{\log L}=2\sqrt{29}\biggl(\frac{\log x}{\log\log\log x}\biggr)^{\frac12}\tag{6.71}$$

so that

$$\log u=(\tfrac12+o(1))\log\log x\quad\text{and}\quad\log\log u=\log\log\log x+o(1)\tag{6.72}$$

as $x \to \infty$. By the definitions of sets $\mathscr{B}^{(i)}$, we have

$$\#\mathscr{B}(x,a) = \sum_{i=1}^{7} \#\mathscr{E}^{(i)} + \#\mathscr{B}^{(7)}.$$

By Claim 6.1, Claim 6.3, Claim 6.4, Claim 6.5, Claim 6.8, we have

$$\begin{aligned}&\#\mathscr{E}^{(1)}, \#\mathscr{E}^{(3)}, \#\mathscr{E}^{(4)}, \#\mathscr{E}^{(5)}, \#\mathscr{B}^{(7)}\\ &\ll x^{\frac{1}{2}}(\log x)^6 L^{-1} \ll x^{\frac{1}{2}} \exp\big(-(\tfrac{1}{2\sqrt{29}} + o(1))(\log x \log\log\log x)^{\frac{1}{2}}\big)\end{aligned}$$

as $x \to \infty$. By (6.71), (6.72) and Claim 6.2, we have

$$\begin{aligned}\#\mathscr{E}^{(2)} &\ll x^{\frac{1}{2}} \log x \exp\big(-\tfrac{1}{16} u \log u\big)\\ &\ll x^{\frac{1}{2}} \exp\left(-\left(\frac{\sqrt{29}}{8} + o(1)\right)\left(\frac{\log x}{\log\log\log x}\right)^{\frac{1}{2}} \log\log x\right)\\ &\ll x^{\frac{1}{2}} \exp\big(-\tfrac{1}{2\sqrt{29}}(\log x \log\log\log x)^{\frac{1}{2}}\big)\end{aligned}$$

as $x \to \infty$. By Claim 6.6 and $a \le \frac{\log x}{\log\log x}$,

$$\begin{aligned}\#\mathscr{E}^{(6)} \ll_\varepsilon 2^{\frac{a}{2}} x^{\frac{1}{2}-\varepsilon} &= x^{\frac{1}{2}} \exp\left(-\varepsilon \log x + \frac{\log 2}{2} \frac{\log x}{\log\log x}\right)\\ &\ll x^{\frac{1}{2}} \exp\big(-(\tfrac{1}{2\sqrt{29}} + o_\varepsilon(1))(\log x \log\log\log x)^{\frac{1}{2}}\big)\end{aligned}$$

as $x \to \infty$. Finally, by (6.71), (6.72) and Claim 6.7,

$$\begin{aligned}\#\mathscr{E}^{(7)} &\ll x^{\frac{1}{2}}(\log x)^2 \exp\big(-\tfrac{1}{4\cdot 29}(1-48\varepsilon) u \log\log u\big)\\ &\ll x^{\frac{1}{2}} \exp\big(-\tfrac{1}{2\sqrt{29}}(1-48\varepsilon + o(1))(\log x \log\log\log x)^{\frac{1}{2}}\big)\end{aligned}$$

as $x \to \infty$. Combining the above estimates, we have

$$\sum_{1 \le a \le \frac{\log x}{\log\log x}} \#\mathscr{B}(x,a) \ll x^{\frac{1}{2}} \exp\big(-\tfrac{1}{2\sqrt{29}}(1-48\varepsilon + o_\varepsilon(1))(\log x \log\log\log x)^{\frac{1}{2}}\big)$$

as $x \to \infty$. Since $\varepsilon \in (0, \frac{1}{48})$ is chosen arbitrarily, we arrive at the theorem. □

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